THE MATHEMATICAL ANALYS

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SOME QUEUING PROBLE

By

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C E R T I F I C A T E

This is to certify that the thesis entitled 'The Mathematical Analysis of Some Queuing Problems' by Shri S.K. Gupta for the award of the Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried out by him under my supervision and guidance for the last three years. The thesis has, in my opinion, reached the standard fulfilling the requirements of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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P R E F A C E

Delays in giving service occur whenever in any process or system, the momentary demand exceeds the available capacity of the serving personnel or equipment. Familiar examples are people waiting at the ration shop, at the cinema ticket window and in a doctor's clinic. Automobiles waiting to cross arterial highways and toll bridges, airplanes seeking to land at busy airports, and ships awaiting dock space to unload are examples of costly and critical delay situations in the transport field. In industry there are such diverse waiting line problems as the flow of materials and machinery to assembly and test positions, the extent to which inoperative machines should be required to wait for repairs, goods lying in a store as inventory, and the economic backlog of typing work in a centralized stenographic department. In all of these the waiting 'customers' comprise a queue. Queuing theory is the study of the formation of such customer waiting lines and the characteristics of the delays they experience. It must of course, however, be admitted that the assumptions made in the analysis of the preponderant number of applicatio of queuing theory have yet to be justified in many cases.

Queuing theory is supposed to be a branch of a broad discipline of knowledge, called 'Operations Research'. However, the author believes that on honesty considerations it should be regarded as a branch of 'Applied Probability Theory'. No doubt that when we consider some queuing system, we have in our minds some optimisation problem, but the actual result, in many cases, that can actually be applied is so particular a case that the prob

looses its meaning altogether. On the other hand, in its most generality also, the system may be regarded as a good problem in applied probability theory. This is true of the problems considered in this thesis also.

As we are all aware, many gaps exist today, as in anyother science, in the literature on queuing theory. The purpose of the present thesis is to attempt to fill in some of these gaps by exploring some relatively unexplored fields of the subject. Strangely enough, of course not very strangely, while this process of the filling in of gaps has helped in solving some problems, it has given rise to a number of other interesting, difficult and even challenging problems.

The present thesis consists of six chapters. Chapter I is a general introductory chapter and in particular it contains a brief review of the up-to-date literature as also the problems considered in the subsequent chapters.

The subject matter of the present thesis (chapters II to VI) is compiled from the following papers written by the author during the past three years (Papers marked with an asterisk are in collaboration with Mr. J.K. Goyal).

<u>Chapter II</u>

1. Queues with Mixed-Erlangian input and exponential service time distribution with finite waiting space.

   (Canadian Operations Research Journal, Journal of the Operations Research Society of Canada, Vol. 3, March 1965, pp. 22-28)

2. Queues with Poisson input and Mixed-Erlangian service time distribution with finite waiting space.

   (Accepted for publication in the Journal of the Operations Research Society of Japan)

Chapter III

*1. Queues with Poisson input and hyper-exponential output with finite waiting space.

(Operations Research, Journal of the Operations Research Society of America, Vol. 12, No. 1, 1964, pp. 75-81.)

*2. Queues with hyper-Poisson input and exponential output with finite waiting space.

(Operations Research, Journal of the Operations Research Society of America, Vol. 12, No. 1, 1964, pp. 82-86.)

3. Queues fed by Poisson input and hyper-mixed Erlangian service time distribution with finite waiting space.

(Unternehmensforschung, Organ der Deutschen Gesellschaft fur Unternehmensforschung und der Osterreichischen Fachgruppe fur Unternehmensforschung Unter Mitwirkung der Schweizerischen Vereinigung fur Operations Research, Band 9, Heft 2, 1965, pp.80-90.)

4. Queues fed by hyper-mixed-Erlangian input and exponential service time distribution with finite waiting space.

(To be published).

5. Queues with hyper-general service time distributions.

(Canadian Operations Research Journal, Journal of the Operations Research Society of Canada, Vol. 3, No. 2, 1965, pp. 90-95.)

Chapter IV

1. Queues with Poisson input and hyper-exponential service time distribution with state dependant arrival and service rates.

(To be published).

2. Queues with hyper-Poisson input and exponential service time distribution with state dependant arrival and service rates.

(To be published).

Chapter V

1. Queues with Batch Poisson arrivals and a general class of service time distributions.

(Journal of Industrial Engineering, Journal of the Institute of Industrial Engineers of America, Vol. XV, No. 6, 1964, pp. 319-320)

*2. Queues with batch Poisson arrivals and hyper-exponential service.

(Accepted for publication in Metrika, Zeitschrift fur theoritische und angewandte Statistik, Germany.)

3. On bulk queues with state dependant arrival and service rates.

(To be published).

Chapter VI

1. Analysis of a two-channel queuing problem with ordered entry.

(Accepted for publication, Journal of Industrial Engineering, Journal of the Institute of Industrial Engineers of America).

In addition to these main six chapters, the thesis also contains an Appendix which includes the numerical work from which graphs appearing in the main thesis have been drawn. The bibliography is appended at the end.

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C O N T E N T S*

* For abbreviations used in this table, see page 11.

CHAPTER I

INTRODUCTION

The world today is experiencing a population explosion and some of the more important problems accruing out of this overpopulation are the food and housing problems, the enormous demand of service in any system and the ability of the people to live with this increased congestion. Almost every day each one of us has to spend some time while standing in a queue. If this time is calculated in hours over the life of any individual, the number representing it will simply be appalingly large. Therefore, it was realised some time ago that we develop some theoretical tools to check this congestion.

Queuing theory is one such tool which originated initially in the beginning of this century to reduce the telephone congestion. However, now the theory has reached a stage that it could be applied to any situation where congestion is involved, the congestion may be due to animate beings or inanimate things.

In the present thesis we propose to investigate the mathematical of some queuing systems.

The Queuing Problem

Any queuing system, as observed by Kendall (1951), is specified following three characteristics:

i) Arrival Pattern,

ii) Service Pattern,

iii) Queue Discipline.

Given these three characteristics, we can proceed mathematically in the hope of finding out the following:

i) System (or queue) length distribution,

ii) Waiting time distribution,

iii) Busy period distribution.

We shall now make precise mathematical statements about the terms used so far.

Arrival Pattern (Input Distribution): Suppose that the units arrive at a service facility at the instants $t_0, t_1, \ldots\ldots, t_n, \ldots\ldots\ldots$ of time and let $T_n = t_n - t_{n-1}$ be the time gap between the arrival of nth and (n+1)st units. Let $t_0 = 0$ and the inter-arrival periods T_n $(n = 1,2,\ldots.)$ be a sequence of independently and identically distributed non-negative random variables with common distribution function, say $A(x)$. In such a case the process $\{t_n\}$ is termed as a Renewal Process and we say that $A(x)$ represents the input distribution or the inter-arrival time distribution.

Service Pattern (Service Time Distribution): Suppose that the service duration of the nth unit in the arrival sequence is s_n $(n = 1,2,\ldots.)$. Let the elements of the sequence of service times $\{s_n\}$ be independently and identically distributed non-negative random variables with common distribution function, say $B(x)$. This $B(x)$ is said to represent the service time distribution

The sequence of service times, $\{s_n\}$, is also usually supposed to be independent of the input process $\{t_n\}$. Moreover, it is assumed that each of the distribution functions, $A(x)$ and $B(x)$, are absolutely continuous and that each of the expectations

$$M_1 = \int_0^\infty x \, dA(x), \qquad M_2 = \int_0^\infty x \, dB(x)$$

exist.

Queue Discipline: The queue discipline specifies the particular order in which the incoming units are to be served. The usual queue discipline, assumed in most of the models investigated so far, is 'first come, first served'. In practice, however, we also come across situations in which the queue discipline is 'last come, first served', e.g. in a big godown, the items which come last are taken out first. An extremely difficult queue discipline to handle might be 'random selection' or 'might is right'. However, we will always assume that the queue discipline is 'first come, first served', i.e. service is in order of arrival.

System Length Distribution: Let $N(t)$ denote the number of units waiting or being served at time t. The probability distribution of $N(t)$ is called the system length distribution.

If $N(t) = k$, we say that the system is in state E_k at time t. Thus $N(t) = 0$ means that the system is in state E_0, i.e. the facility is idle and $N(t) = 1$ means that the system is in state E_1, i.e. there is no unit in the queue (only service facility is occupied).

Queue Length Distribution: Let $H(t)$ denote the number of units waiting at time t, i.e. the number of units present in the system at time t excluding the one undergoing service, if there be any. The probability distribution of $H(t)$ is called the queue length distribution.

It may, however, be noted that given the system length distribution, we can calculate from it the queue length distribution, and vice versa.

Waiting Time Distribution: Let w_n be the waiting time of the nth unit. Then $F_n(x) = \text{Prob.}\ (w_n \leqslant x)$ represents the distribution function of the waiting time distribution of the nth customer.

Virtual Waiting Time Distribution: Let $X(t)$ denote the length of time that a unit arriving at time t will have to wait before entering service. It may be observed that this is not necessarily the waiting time of any particular unit, since no unit might have arrived at time t. Instead it is called the virtual waiting time and simply waiting time if no confusion is likely to be there. It is defined exclusive of the service time. The probability distribution of $X(t)$ is called the virtual waiting time distribution.

The virtual waiting time has another interesting interpretation as follows: $X(t)$ is the time at the instant t needed to complete the serving of all the units present in the queue. With this interpretation $X(t)$ is termed as the Occupation Time of the server at the instant t.

Busy Period Distribution: Let $Y(t)$ be the time required for the system to be in state E_0 for the first time when initially it was in state E_1. Then the probability distribution of $Y(t)$ is called the busy period distribution.

If the system starts with $i>1$ initial units, then we expect intuitively that the busy period will be different from what it is when $i = 1$. This particular busy period is called the initial busy period distribution. But knowing the busy period distribution, say $C(x)$, as defined above, we can express the initial busy period distribution in terms of $C(x)$.

It is quite obvious, due to the nature of fluctuations in the inter-arrival times and service times, that $N(t)$, $H(t)$, $X(t)$ and $Y(t)$ are random variables depending upon time and are thus stochastic processes. The success in solving any queuing problem usually depends upon our ability to extract Markov processes out of these stochastic processes.

The general queuing problem, as usually meant today, is as follows:

Given the triplet

(Arrival time distribution, Service time distribution, queue discipline)

to find the triplet

(System length distribution, waiting time distribution, busy period distribution).

Some Other Assumptions

To simplify the mathematical analysis, we have already made some assumptions, e.g. the members of sequences $\{T_n\}$ and $\{s_n\}$ are assumed to be independently and identically distributed non-negative random variables with common distributions $A(x)$ and $B(x)$ respectively. Now we make some assumptions about the instants when the units enter for service. We suppose that if a unit arrives to find the service facility idle, then its service commences immediately, and if it arrives to find the service facility busy and n units waiting, then it waits at the spot specified by the queue discipline until the service facility is idle and no unit is waiting before it, then its service commences immediately, i.e. the time gap between finishing the service of one unit and starting service on the next is zero if the latter unit had to wait.

It may be of interest to note here that under the assumptions made above the quantity that is effected by changing the queue discipline is $X(t)$. As expected, the queue discipline 'first come, first served' has received the maximum attention of the workers because of the resulting mathematical simplicity. Among other queue disciplines, 'random selection for service' and 'last come, first served' have also received some attention.

Although Moller (1942) was the first to study the queue discipline 'random selection for service', it was Vaulot (1946) who gave an exact formulation of the problem. Later, however, Pollaczek (1959), Palm (1957)

and Riordan (1953) attempted to solve the problem formulated by Vaulot (1946). Some useful computations were given by Wilkinson (1953), and Le Roy (1957) used matrix methods to study the problem. Kingman (1962a) has also studied such a problem.

The queue discipline 'last come, first served' has also been considered by Vaulot (1954) and Wishart (1960).

Types of Queuing Problems

Broadly speaking, we can distinguish between two types of common characteristics of queues as follows:

i) deterministic characteristics,

ii) probabilistic characteristics.

In case (i), we know with certainty the moments of time at which the units come, the exact duration of their service times and some well defined queue discipline. It is quite obvious then that ordinary arithmetic can be used to solve such a queuing problem, and as such it is not of much interest to us. It may, however, be remarked that even in such a simple case, the calculations may become prohibitive, as for example the exercises appended to the illustration of a deterministic queue given by Saaty (1961).

In case (ii), however, we know only the probabilities that the units come between such and such intervals of time, their respective service times are known probabilistically, and, of course, some well defined queue discipline. Obviously then we can get the results also in terms of probability only. Thus we cannot calculate the 'queue length' at any time, we can only find the 'probability that the queue length at any time is this much' or the 'expected queue length at any time'.

Some Probability Distributions

Below we explain some of the arrival and/or service time distributions with which we shall be concerned with in this thesis.

(i) Poisson Process:

An infinite sequence of independent events, each occurring at an instant of time, forms a Poisson process if

(a) the total number of events happening in any time interval, x, does not depend upon any of the events which occurred before the beginning of the interval x.

(b) the probability of the happening of an event in any time interval, δx, is $\lambda \delta x + 0[(\delta x)^2]$, λ being a constant.

The importance of Poisson processes in queuing theory is because of the fact that arrival and service moments are random and therefore Poisson processes fit in naturally in a number of queuing situations. The following two impdrtant facts about Poisson processes are worth noting and we shall make an extensive use of these in the present thesis.

(1) For a Poisson process the number of events occurring in an interval t is a random variable which follows a Poisson distribution with parameter λt, λ being a constant, i.e.

the chance of n events in time t

$$= \frac{(\lambda t)^n e^{-\lambda t}}{\underline{|n}} \qquad (1)$$

(2) For a Poisson process the intervals of time between successive events follow an exponential distribution, i.e.

$$\text{Prob.}\ (u \leqslant T \leqslant u + du) = \lambda e^{-\lambda u}\, du. \qquad (2)$$

The exponential distribution, given by (2), has an important property, called the 'forgetfulness property'. When this distribution is used as a service time distribution, the forgetfulness property states that the 'probability at any time of a unit's service being finished in some interval of time is independent of how long the service has already been performed'.

A number of other interesting properties of Poisson processes are derived in Hadley and Whitin (1963).

(ii) <u>Regular Distribution:</u>

The density function is given by

$$f(t) = \delta(t - a) \tag{3}$$

where a is a constant and δ is the Dirac delta function. This is the distribution in the deterministic case where the time interval has a constant value a.

(iii) <u>Erlangian Distributions:</u>

The density function is given by

$$f(t) = \frac{(\lambda)^k \, t^{k-1} \, e^{-\lambda t}}{\underline{|k-1}} \tag{4}$$

where $k \geqslant 1$ is an integer and $\lambda > 0$ is a real number.

It can be proved that the k-Erlang distribution is the distribution of a random variable which is the sum of k independent random variables, all these k random variables being distributed according to the same exponential distribution. Another interpretation of this distribution is through phases. An event is supposed to happen if it passes through each of k phases, one after the other, and the time of stay in each phase is distributed according to the same negative exponential distribution with parameter λ.

Obviously, when $k = 1$, it reduces to the exponential distribution and when $k \longrightarrow \infty$, it reduces to the regular distribution. Thus the Erlangian distributions fill in some of the gaps between the two extreme distributions, viz. the regular and exponential distributions.

(iv) <u>Mixed-Erlangian Distributions of the First Kind:</u>

The density function is given by

$$f(t) = \sum_{r=1}^{k} \frac{c_r e^{-\lambda t} (\lambda t)^{r-1} \lambda}{\lfloor r-1}\tag{5}$$

where k and λ are as in (4) and c_r are probabilities satisfying the constraints

$$0 \leqslant c_r \leqslant 1, \qquad \sum_{r=1}^{k} c_r = 1.$$

This distribution is a generalisation of the Erlangian distribution in the sense that the event is not necessarily to pass through each of the k phases. Instead, it may pass through r ($r = 1,2,\ldots,k$) phases with probability c_r. Thus when $c_r = \delta_{rk}$, it reduces to the k-Erlang case. Thus this distribution may be thought of as an Erlangian distribution, say E_n, where n itself is a random variable with probability density given by, prob. $(n = r) = c_r$, and n could assume the values $n = 1,2,\ldots,k$.

(v) <u>Mixed-Erlangian Distributions of the Second Kind:</u>

The density function is given by

$$f(t) = \left(\prod_{i=1}^{k} \lambda_i\right)\left(\sum_{i=1}^{k} \frac{e^{-\lambda_i t}}{\prod_{\substack{j=1 \\ j\neq i}} (\lambda_j - \lambda_i)}\right)\tag{6}$$

where k is as in (5) and all $\lambda_i > 0$ are real numbers.

This distribution is a generalisation of the Erlangian distribution in the sense that the distributions for the time of stay in various phases have

different parameters, i.e., the distribution for the time of stay in the rth phase is exponential with parameter λ_r.

(vi) Mixed-Mixed Erlangian Distributions:

The density function is given by

$$f(t) = \sum_{r=1}^{k} \left[c_r \left(\prod_{i=1}^{r} \lambda_i \right) \left(\sum_{i=1}^{r} \frac{e^{-\lambda_i t}}{\prod_{\substack{j=1 \\ j \neq i}}^{r} (\lambda_j - \lambda_i)} \right) \right] \quad (7)$$

where k and c_r are as in (5) and all $\lambda_i > 0$ are real numbers.

Thus this distribution is a generalisation of (6) in the same sense as (5) is a generalisation of (4).

(vii) Hyper-Exponential Distribution:

The density function is given by

$$f(t) = \sum_{r=1}^{n} \sigma_r \mu_r e^{-\mu_r t} \quad (8)$$

where $n \geqslant 1$ is an integer, μ_r and σ_r are positive real numbers such that

$$\sum_{r=1}^{n} \sigma_r = 1.$$

We explain this distribution as a service time distribution.

Suppose we have n branches at the service facility with different service rates. A unit enters the rth branch a fraction σ_r of the time on the average. If the service time distribution in the rth branch is exponential with parameter μ_r, then the overall service time distribution is as in (8).

Similarly, we can talk of hyper-general distribution, if the service time distribution in the rth branch has density function $D_r(x)$.

Thus this distribution may be used when a single operator is supposed to discharge units requiring various types of services, the number of branches in the service facility being equal to the types of services offered at the facility.

Notations

We will use the following notations for the various distributions explained above:

Distribution	Notation
Regular	D
Exponential	M
Poisson	M
k-Erlang	E_k
Mixed-k-Erlang (ith kind)	$iM\text{-}E_k$, $i = 1,2$
Mixed-Mixed-k-Erlang	$M\text{-}M\text{-}E_k$
Hyper-exponential or Poisson with n branches	$H\text{-}M_n$
Hyper-mixed-k-Erlang with n branches	$H\text{-}1M\text{-}E_{nk}$
General Input	GI
General Service time distribution	G
Hyper-General with n branches	$H\text{-}G_n$
Hyper-constant with n branches	$H\text{-}D_n$

To abbreviate the queuing systems, we will follow Kendall (1951), i.e. a queuing system will be written as:

Input distribution/Service time distribution/number of servers.

Thus a queuing system with Poisson input, regular service and r servers will be denoted by M/D/r.

Techniques Used To Solve the Problems

Now we describe some of the techniques that have been used in queuing theory to solve the various problems.

(1) Differential Difference Equations Technique:

It is proved in the theory of Markov processes that the transition probabilities $p_{ik}(t)$ satisfy the so called Chapmann-Kolmogorov equations

$$p_{ik}(t+s) = \sum_{j=1}^{n} p_{ij}(t)\, p_{jk}(s) \qquad (t>0,\ s>0) \qquad (9)$$

$$(i,k = 1,2,\ldots,n)$$

expressing the fact the the passage from state i to state k during time t + s must occur via some intermediate state j at time t, i.e. it expresses the consistency condition for the transition probabilities.

Many queuing problems have been solved by using the above Chapmann-Kolmogorov equations. In queuing problems the process studied is either itself Markovian or even if it is not, we can render it Markovian by the techniques mentioned below. Once the process under consideration is rendered Markovian, we may write down the corresponding Chapmann-Kolmogorov equations, which are differential-difference equations, and then proceed to solve these equations by any method we like, typically the generating functions technique.

(2) Supplementary Variables Technique:

Suppose we are interested in some stochastic process $X(t)$, which is not Markovian. Then we can always find a sufficient number of variables, say $Y_i(t)$, so that the joint process $[X(t),\ Y_1(t),\ Y_2(t), \ldots\ldots]$ becomes Markovian. The variables $Y_i(t)$ are called supplementary variables. Clearly the process becomes many dimensional and the analysis becomes quite intriguing, usually depending upon the number of supplementary variables required to render the original process Markovian. Once the process is rendered Markovian, we can write down the corresponding Chapmann-Kolmogorov equations, and proceed to solve these equations, if possible.

This technique was called 'augmentation technique' by Kendall (1951) and 'supplementary variable technique' by Cox (1955) who used this technique to study the steady state case of the queuing system M/G/1. Later, Keilson and Kooharian (1960) and Wishart (1960) used this technique to study the transient case of the queuing system M/G/1. The method itself also appears in Vaulot (1927), and in Koesten (1948-49).

To explain the use of this technique in queuing theory, consider the queuing system M/G/1. If we are interested in the statistical variation of N(t), the number of units in the system at time t, Keilson and Kooharian (1960) used the supplementary variable as 'the elapsed service time, say X(t), of the unit undergoing service'. Wishart (1960) used the supplementary variable 'remaining service time of the unit in service, say y(t)'. Further illustrations of the method may be found in a paper on the M/G/1 queuing system by Heathcote (1961), the GI/G/1 study of Keilson and Kooharian (1962), the two priority queuing studies of Jaiswal (1961a, 1962a) and the bulk queuing study of Keilson (1962).

(3) Phase Technique:

Usually queuing systems with general arrival and/or service time distributions are difficult enough to be tackled. Phase technique is a way to cope with this complexity. We approximate the given distribution by the Erlang E_k distribution, given by the k-fold convolution of the exponential distribution with itself. The parameter k is to be so chosen as to control the variance as well as the mean. We may note that while using this technique, we are required to use both the techniques mentioned above.

An inter-arrival period or service having this distribution may be regarded as consisting of k successive phases in each one of which the time spent by the unit has the same exponential distribution. This technique was further generalised by Gaver (1954) and Luchak (1956) in the sense that

every unit is not required to pass through k phases. Instead, they introduced the probabilities c_r, that a unit requires service only in the first r, $1 \leqslant r \leqslant k$, successive phases, i.e. the number of phases through all of which a unit has to pass itself being a random variable..

In the present thesis we will use only these three techniques to solve the various problems posed.

(4) Imbedded Markov Chain Technique:

The stochastic processes associated with any queuing system, except when both the inter-arrival and service time distributions are exponential, are usually non-Markovian. The supplementary variables technique, already described, is one method to render the process Markovian by the inclusion of some supplementary variables. Another techniques, called 'imbedded Markov Chain' technique has been developed by Kendall (1951, 1953) to deal with some of these problems.

Consider, for example, the queuing system M/G/1. The stochastic processe $N(t)$ representing the number of units present in the system at time t, does not constitute a Markov process, except in the very particular case M/M/1. However, if we consider the set Z of those moments when a service is completed then $N(t)$ when $t \in Z$ constitutes a Markov chain, because the input is of Poisson type. Obviously then the Markov chain $N(t)$, $t \in Z$, is imbedded in the stochastic process $N(t)$, $t \in (-\infty, +\infty)$. Once such a Markov chain is constructed, we can proceed with its analysis by the methods given in Feller's book (1950).

This technique has also been used by Bailey (1954) and Downton (1955, 1956) to study some bulk service queuing problems.

(5) <u>Some Other Techniques</u>:

In addition to the techniques discussed above, one may also include the following techniques:

(i) <u>Extended Markov Chain Method</u>: Gaver (1959) studied the queuing system M/G/1 by considering the sequence of times when service starts on a new unit and called it the 'extended Markov chain method'. Priority queues have also been treated by Gaver (1962) by this method.

(ii) <u>Semi-Markov Method</u>: Fabens (1961) observed that for the queuing system M/G/1, the number of units H(t) at time t, in queue at the instant after service initiation preceding time t is a semi-Markov process, and discussed this queuing system as such.

(iii) <u>Combinatorial Methods</u>: The queuing systems M/G/1 and GI/M/1 have also been treated by Prabhu (1960) and Takacs (1962a, 1962b, 1962c) by using combinatorial methods.

By a perusal of the queuing problems tackled so far, one finds that a central analytical problem is reached by almost all methods requiring for example the use of generating functions and Laplace transforms and the formal solutions thus obtained are usually given in the transform space involving roots of some transcendental equations. It may be mentioned here that the process of going back to the original space from the transform space in most of the cases is tedious enough and as such the virtue of such formal solutions may be questioned. Still, however, in the absence of the availability of better procedures, we have to be contented with whatever possible we can do with the procedures available.

Steady State Solutions

In general, we are interested in finding out the probabilities of interest in queuing theory as functions of time. But, however, as remarked above, we can only find the Laplace transforms of these probabilities for the more complex queuing systems and the process of inverting these transforms is tedious enough and is, therefore, sometimes avoided. To avoid this difficulty, very often, we consider only the steady state solution, i.e. we find out the limit of the probabilities as the time, t, tends to infinity. It may be remarked here that this limit may not exist for all values of the arrival and service parameters and therefore we are required to find out the conditions under which the steady state solution is possible. A typical condition like this was found by Lindley (1952) for the queuing system GI/G/1. Lindley (1952) proved that whenever the traffic intensity is less than unity for GI/G/1, the steady state solution exists. It is also interesting to observe here that this condition is true when the arrival and service parameters are constant. If, for example, these parameters depend upon the state of the system then steady state solution may exist even when the traffic intensity is equal to unity, see for example Cox and Smith (1961, p. 50).

The Concept of Limited Waiting Space

In most of the queuing situations considered so far no attention was paid to the fact that the waiting space for units may be limited. In general the formulae for queuing systems with limited waiting space are more complicated, but this complication is compensated by the fact that such a queuing system is more general in the sense that we can deduce from it the corresponding formulae when infinite waiting space is allowed or when no queue is allowed. Another reason for studying such systems is that steady state solutions always exist regardless of the value of the traffic intensity.

In most of the queuing problems considered in this thesis, we assume that only a limited waiting space is available.

Problems Considered in this Thesis

We now give below a brief description of the contents of the various chapters of this thesis as also the relevant literature.

Chapter I is divided into two sections. In section I, we discuss the queuing system $M-M-E_j/M/1$ and in section II, the queuing system $M/M-M-E_j/1$ is discussed.

The problems tackled in this chapter illustrate the use of phase technique in queuing theory which may be traced back to A.K. Erlang (see Brockmeyer et. al., 1948). This technique was generalised by Gaver (1954) who preferred an infinite number of phases instead of a finite number and introduced the probabilities c_r that a unit requires service in r phases only. Thus if the service spent in each phase is distributed exponentially with the same parameter μ, the overall service time distribution is given by

$$s(t) = \sum_{r=1}^{\infty} c_r e^{-\mu t} \frac{(\mu t)^{r-1} \mu}{\lfloor r-1} \qquad (10)$$

subject to the conditions

$$\sum_{r=1}^{\infty} c_r = 1, \qquad 0 \leqslant c_r \leqslant 1.$$

By giving special values to c_r, Gaver (1954) obtained a number of distributions.

It was, however, Luchak (1956) who pointed out that for a general s(t), the value of c_r when calculated by using the Taylor's expansion, i.e. from

$$\mu^r c_r = \left[\frac{d^{r-1}}{dt^{r-1}} \; s(t) \; e^{\mu t}\right]_{t=0} \qquad (11)$$

might violate the conditions $\sum_{r=1}^{\infty} c_r = 1, \quad 0 \leqslant c_r \leqslant 1$ for $\mu > 0$.

To resolve this difficulty he suggested that we should still have a finite number of phases, as in Erlang's distribution. Thus, extracting some ideas from Erlang's distribution and some from those suggested by Gaver (1954), i.e., the concept of the probabilities c_r, he considered the overall service time distribution as

$$s(t) = \sum_{r=1}^{j} c_r \, e^{-\mu t} \frac{\mu(\mu t)^{r-1}}{\lfloor r-1}$$ (12)

where

(i) $0 \leq c_r \leq 1$, for $r = 1,2,\ldots,j$,

(ii) $\sum_{r=1}^{j} c_r = 1$.

But one defect in his study was that he could determine the distribution of the number of phases only instead of the number of units in the system or queue.

This difficulty was overcome by Jaiswal (1961) who studied the queuing system $1M\text{-}E_j/M/1$ and defined the probabilities $p_{n,r}(t)$, that at time t there are n units in the system and the unit in the arrival channel being in the rth phase. He thus fused together the phase technique and the supplementary variable technique to resolve the difficulty encountered in the study of this problem. Following Jaiswal (1961), Jain (1962) studied the queuing system $M/1M\text{-}E_j/1$ and determined the system size distribution and the Laplace transform of the waiting time distribution in the steady state case.

The question now arises as to why should we have the same exponential distribution for all the phases. Surely we could have some distribution other than the exponential or various distributions for the various phases. The problems considered in chapter II of this thesis are an answer (partial) to the question raised above.

Thus we study in chapter II the problem considered by Jaiswal (1961) and by Jain (1962) when the parameters of the distributions for the various phases are different. This introduction of the different parameters facilitates the simulation of some other distributions and also expands the field of applications to cases where the mean times of stay of the units in the various phases of the arrival (or service) channel are not same.

Chapter III is divided into three sections. In section I, the queuing system $M/H\text{-}1M\text{-}E_{nk}/1$ is discussed, in section II, the queuing system $H\text{-}1M\text{-}E_{nk}/M/1$ is discussed and in section III, the queuing system $M/H\text{-}G_N/1$ is discussed. Results for the queuing systems $H\text{-}M_N/M/1$, $M/H\text{-}M_N/1$ and $M/H\text{-}D_N/1$ are deduced as particular cases.

In addition to the Erlang's method and its extensions described above, Morse (1958), in his book has discussed another simulating distribution which he calls 'hyper-exponential'. This distribution again is obtained as a mixture of the exponential distributions. Saaty (1961) in his book has pointed out that hyper-exponential distribution can be used to simulate any distribution representable by a completely monotonic function.

Morse (1958) has studied the steady state case of the queuing systems $H\text{-}M_2/M/1$ and $M/H\text{-}M_2/1$ when an infinite queue is allowed. But two branches may not suffice to simulate all distributions representable by completely monotonic functions. Thus, Gupta and Goyal (1964a, 1964b) studied the steady state case of the queuing systems $H\text{-}M_N/M/1$ and $M/H\text{-}M_N/1$ when only a finite queue is allowed and evaluated the system size distributions in both the cases.

However, once again we may raise the question whether or not we can use some distributions other than the exponential for the various branches of the arrival (or service) channel. A partial answer to this question was

provided by the author (1965a, 1965b, 1965c) and thus he studied the queuing systems $M/H{-}1M{-}E_{nk}/1$, $M/H{-}G_{M}/1$ and $H{-}1M{-}E_{nk}/1$. From the results obtained this way, the results arrived at by Gupta and Goyal (1964a, 1964b) can be deduced as particular cases. These problems are presented in chapter III.

Chapter IV is also divided into two sections. In section I, we consider the queuing system $M/H{-}M_{k}/1$ and in section II the queuing system $H{-}M_{k}/M/1$ is discussed when the arrival and service parameters depend upon the state of the system, the functions defining this dependance being assumed to be arbitrary.

Queues with state dependant arrival and/or service parameters are encountered in a number of situations and various workers have considered such queues. Thus the effect of balking and/or reneging is to introduce state dependant parameters. Also if one considers a queuing system, in which the source from which the units join the queue or service, contains a finite number of units, one is essentially studying a queuing system with state dependant parameters. As an example of this one may consider the 'machine interference problem'. Many server queuing processes have the effect of introducing state dependant parameters.

Queues with balking or reneging have been considered by Haight (1957, 1959, 1960) and those with balking and/or reneging by Ancker and Gafarian (1963, 1963a, 1963b). Some other workers who have considered queues with state dependant parameters are Hiller, Conway and Maxwell (1964); Conway and Maxwell (1961); Vetaw and Stower (1956); Jaiswal and Thiruvengadam (1963); Thiruvengadam (1964); Thiruvengadam and Jaiswal (1964, 1964a, 1963) etc. Eisen (1963) has considered a queuing problem in which the service parameter is a function both of time and the state of the system.

But in all these studies, one finds that the functions defining the dependance of the arrival and/or service parameters upon the state of the system are particular types of functions. The author (1965d, 1965e) has considered queuing systems $M\text{-}H_k/M/1$ and $M/H\text{-}M_k/1$ with state dependant parameters with arbitrary functions. The results of this study are presented in chapter IV.

It may be remarked here that while studying $M/H\text{-}M_k/1$ with state dependant parameters, the author found that it is possible to interpret the problem as a 'k-type stoppage machine interference problem' by choosing particular forms of the state functions. Earlier, however, Jaiswal and Thiruvengadam (1963) had solved a generalisation of the 'two-type stoppage machine interference problem' posed by Benson and Cox (1951). This remark supports the plea that in addition to their simulating character, hyper-distributions have applications in practical fields also.

Another reason to study queues with state dependant parameters is that, in some sense we can relax the assumption that the elements of the sequences of inter-arrival times and service times are distributed independently of each other.

Still another interesting feature of this study is that we have to search for new methods of attacking the problems because the usual method of generating functions can no more be applied. It is interesting to note that the heuristic types of techniques used to solve the problems of this chapter are better than the method of generating functions in the sense that now we get recurrence relations connecting the various probabilities instead of a formal solution involving the roots of a polynomial of degree equal to the number of branches in the arrival (or service) channel. Thus, numerical calculations become much more easier with the techniques employed in this chapter.

Chapter V is also divided into two sections. In section I, we start with the queuing system $M/H\text{-}1M\text{-}E_{nk}/1$ in which the units arrive in batches of variable sizes, and deduce the results for the queuing systems $M/H\text{-}M_k/1$ and $M/1M\text{-}E_k/1$. In section II, we discuss the queuing system $H\text{-}M_k/M/1$ with state dependant arrival and service rates in which the units are served in batches of fixed size, S, or the whole queue length, whichever is less.

Bulk queuing models have been considered by a number of workers, the first such study being by Bailey (1954). In bulk queuing models, one can think of a number of queue disciplines. A comprehensive list of such disciplines is given by Arora (1964).

Reference to bulk queuing theory may be made to: Foster (1961, 1964); Conolly (1960), Jaiswal (1960a, 1960b, 1961b, 1962), Natrajan (1962), Foster and Nyunt (1961), Foster and Perera (1964), Bhat (1964), Keilson (1962), Miller (1959), Bailey (1954), Downton (1955, 1956), Arora (1964a) etc.

All these authors, except Keilson (1962), have assumed that the size of the batch is fixed. However, in practice, we come across situations in which the batch, particularly the arriving batch, is not fixed. Thus Gupta and Goyal (1964c) considered the queuing system $M/H\text{-}M_k/1$ with variable batch arrivals. Again the author (1964) considered the queuing system $M/1M\text{-}E_k/1$ with variable batch arrivals. Later, however, the author (1965) could combine these two systems and studied the queuing system $M/H\text{-}1M\text{-}E_{nk}/1$ with variable batch arrivals and found that the solution of this queuing system can be deduced from the corresponding system $M/H\text{-}1M\text{-}E_{nk}/1$ with single arrivals, discussed in section I of chapter III. In section I of chapter V, we present this approach to the problem and discuss the details of the results for the queuing systems $M/H\text{-}M_k/1$ and $M/1M\text{-}E_k/1$ with variable batch arrivals.

Moreover, none of the authors listed above has introduced state dependant parameters in bulk queuing models. Thus the author (1965d) considered the queuing system $H-M_k/M/1$ with state dependant parameters and bulk service, the size of the batch being S or the whole queue length, whichever is less. This problem is presented in section II of chapter V.

It may be of interest to note here that with bulk service, we cannot deduce the results for the many server queuing systems even if we study the system with state dependant parameters. This, however, is possible if we are not considering bulk service. Thus, we cannot deduce the results of Arora (1964a) when $\mu_1 = \mu_2$ from the present study.

<u>Chapter VI</u> is devoted to the study of a 'graded two-server queue'.

Many-server queues have been studied by a number of workers, including: Erlang (see Brockmeyer et. al., 1948), Molina (1927), Kendall (1953), Fagen and Riordan (1955), Humbel (1960), Homma (1956), Joffe and Ney (1960), Karlin and McGregor (1958), Keifer and Wolfowitz (1955), Saaty (1960), Ancker and Gafarian (1962), Krishnamoorti (1963), Yuh Ming-i (1959), Koenigsberg (1964), Nishida (1965), Hong (1962) etc.

Palm (1936) and Wilkinson (1931), however, considered a different type of many server queue which is characterised as follows: the servers are arranged in ordered groups, and a demand which can be served is served by a server in the group of lowest order having a server free, i.e. the traffic input to a given service group is the overflow from its immediate predecessor (or predecessors).

The asymptotic solution of the integral equation obtained by Palm (1943) has been obtained by Hellman (1963). Hellman (1964) has also considered the generalisation of Palm's problem when the service time distribution for each of the servers is general. Chang-Ting (1962) has given a direct proof of the generalisation of Palm's assertion when he proved that 'if the intensities

of the incoming currents are the same in each circuit, then the average attenuation rate in each circuit increases with the increase of its assigned code'. Here, circuits correspond to servers, code of the circuit corresponds to the server number and attenuation corresponds to the probability of loss.

Palm (1936) and Wilkinson (1931) confined themselves to the problem when no queue in front of any service group is allowed. It was Disney (1962) who attempted to solve this problem when queues in front of service groups are allowed. As a beginning, he wrote the queue equations for two servers only, assuming that not more than M units can be there with the first server and not more than N units can be there with the second server. However, he could not solve this problem; instead he solved the equations explicitly in two very particular cases, namely, (i) M = 1, N = 1, i.e. problem of Palm (1936), (ii) M = 1, N = 3. Disney (1963) again considered this problem and solved it when M = 1 and N may have any value.

The author (1965), however, could solve these equations with general values of M and N by using the technique of generating functions. Some numerical work has also been done when N = 1. This solution of the problem is presented in chapter VI.

It may be remarked that such a many-server queuing system is important in practical situations also, e.g. in conveyor theory as discussed by Disney (1963). Moreover, not much work seems to have been done in this direction, probably because of the complexity of the problem.

Brief Historical Sketch

The history of queuing theory may be divided into two well-defined periods, viz. (before 1951) and (after 1951). The first period began with the pioneering work of Erlang (see Brockmeyer et. al., 1948) early in this century and lasted until the very important and thought provoking paper of

Kendall (1951) appeared. The principal workers in this period were continental Europeans who were concerned primarily with the telephone traffic problems. Their works appeared in relatively inaccessible journals, and frequently in languages other than English.

After 1951, however, the position changed completely and a spree of papers, written mostly in English language appeared in easily accessible journals. So much so that Saaty's book, which appeared in 1961, contained 910 references. Now the authors estimate is that more than 1500 references can be easily collected. Kendall (1964) while reviewing queuing theory has observed that the rate at which papers on queuing theory are appearing now is one paper per week. In the light of these remarks, it may be difficult to review all these papers. Moreover, till about 1960, Saaty (1961) has reviewed most of the interesting papers. As such, below I mention some of the more important papers which have appeared after 1960 and have not been mentioned elsewhere in this chapter.

Takacs (1962) considered the queuing system M/G/1 and obtained explicit formulae for (i) probability that the server is idle at time t; (ii) initial busy period distribution; (iii) busy period distribution other than the initial; and (iv) virtual waiting time distribution. In the other papers also Takacs (1962a, 1963, 1963a) confined himself to the queuing system M/G/1 and obtained the stochastic law of the busy period, i.e. the probability that the busy period has finite length or consists of a finite number of services, and virtual waiting time distribution by using some elementary and combinatorial methods.

Finch (1961) obtained results for the busy period of GI/G/1 by using a combinatorial lemma due to Spitzer (1956). In 1962, Finch considered the

transient behaviour of the bulk service queuing system with finite waiting space, the service being exactly of r units every time. In 1963, Finch considered the single server queuing system characterised by a recurrent input process and Erlang service times. In 1959, Finch had investigated a generalized single server queuing system in which if on arrival the unit finds the server idle then his service does not commence immediately until a time v_n after his arrival, the sequence of random variables v_n being distributed identically and independently with some common arbitrary distribution function. Later Evans and Finch (1962) solved this problem with Erlang input. Burke (1956) and Reich (1957) had proved that the output process of M/M/1 is again a Poisson process with the same parameters as the input process. Finch (1960), proved that for the M/G/1 queuing system the successive departure intervals are independent in the limit only in the case when service time is negative exponential.

Chang (1963) has obtained the distribution for the inter-departure time for the queuing system GI/G/1 by applying the complex variable theory. Mirasol (1963) has shown that for the queuing system $M/G/\infty$, the output distribution is again Poisson. Moreover, the transient output is a Poisson process of the non-homogeneous type, while the steady state output is homogeneous with a rate equal to the input rate.

Kingman has studied the waiting time of a number of queuing problems. In 1962, while studying the effect of queue discipline on waiting time variance, he proved that the mean waiting time is independent of queue discipline while the variance attains its minimum when customers are served in order of arrival. In 1962a, he considered the queuing system M/G/1 and obtained the waiting time distribution in the steady state case when customers are served in random order. In 1962b, he considered the queuing system GI/G/1 and obtained certain inequalities involving the mean and variance of the waiting time. In 1962c, he

considered queues in heavy traffic, i.e. when the traffic intensity, ρ , is less than unity but very near unity, and proved that when $\rho \longrightarrow 1$, the distribution of $(1 - \rho)w$ tends to negative exponential distribution, w being the waiting time. In 1962d, however, he obtained the busy period distribution for GI/G/1 by using a generalisation of Spitzer's identity, when either of the distributions GI or G, has a rational characteristic function.

By using a method similar to that used by Champernown (1956), Prabhu and Bhat (1963) have obtained transition probabilities for the queue length for the queuing system M/G/1. The method, they show, is applicable to systems with batch arrivals or with balking. Prabhu (1962) has also obtained the distribution of $T = \min \{ n | w_n = 0 \}$, w_n being the waiting time of the nth arrival in the queuing system M/G/1. By applying a duality principle, he has deduced Smith's (1953) theorem for GI/M/1 and has also obtained the waiting time distribution for E_k/D/1.

Leynes (1962) proved the existence of a unique stationary waiting time distribution when $u_n = s_n - T_n$ is a stationary process, s_n and T_n being the inter-arrival and service time sequences. Later, Leynes (1962a) showed that these stationary distributions may sometimes be found and he actually obtained some quantitative results also.

Gaver (1963) has obtained the integro-differential equation for the single server queuing system providing two types of services. He has also included in his study the concept of orientation time, i.e. the time required to change from one service to another.

Yadin and Naor (1963) have evaluated the average queue length and queuing time for a single server queuing system with Poisson input in which the service station is closed down intermittently.

Mader and Phillips (1962) have considered a queuing system in which the number of channels is increased by one whenever the queue reaches a given length N. They have assumed that a fixed number of channels are always working and the channels introduced in addition to this fixed number are cancelled whenever the queue drops to n $(0 \leqslant n \leqslant N - 2)$ and a service is completed.

Jones (1963) has investigated the spectra of some of the linear operators associated with some queuing systems, detailed investigation being for the queuing system M/M/1.

Ghosal (1963), by applying the theory of storage, obtains some results for a single server queuing system in which a customer does not wait more than a fixed time k. Ghosal (1962) has also extended Lindley's technique (1952) to obtain the waiting time distribution in different queues in series, each queue having a server.

Neuts (1964) has considered a general queuing model in which the service times of successive customers form an m-stage semi-Markov process.

Coleman and Giammo (1963) have investigated the problem of servicing a finite population by means of a multiple-channel service unit that is only periodically available.

Eisen and Tainiter (1963) have considered a queuing process in which there are two mean arrival and service rates. Not only does the system randomly change from one mode of service and arrival to another, but units arrive at random and require varying amounts of service. Analytic expressions are obtained for the generating functions, the mean queue length, and the mean waiting time. Several practical illustrations are also given.

Sack (1963) has obtained the queue length distribution for M/M/1 by considering the cumulative probabilities $q_n(t)$ that there are atleast n customers at time t.

In addition to offering some very critical comments on single server queuing theory and reviewing the techniques employed to solve the queuing problems, Keilson (1964) advocates for the wide applicability of Hilbert problem methods to queuing studies. He has also observed a relation between the GI/M/1 system and the M/G/1 system of restricted queue.

The latest authentic review of queuing theory is again given by Kendall (1964), wherein he has also suggested some of the interesting unsolved problems. One such is the following problem regarding the queue output: we observe (say from $t = 0$ to $t = \infty$) the output of a queuing system; how can we tell whether it is of the form GI/G/s and identify inter-arrival and service time distributions.

Koenigsberg (1964), after surveying the relevant two server problems, introduces the concept of jockeying, i.e. the movement of a waiting unit from one queue to another, in three different models and compares the results obtained with those without jockeying.

Some interesting papers on queues in series and networks of queues have also appeared during the period being reviewed. Evans (1964) has considered some queuing models in which jobs require several different services, in any sequence, possibly simultaneously, by a number of servers. In 1965, Evans observed that there is a void between the theory and simulation which can be and should be filled by numerical analysis. Much more likely, the lack of numerical work, he observes, is an indication of the uselessness of theoretical results in design questions and the mathematical simplicity of many of these results.

Friedman (1965) has considered the following two tandem stage queuing systems: Model I — n stages $A_1, A_2, \ldots\ldots, A_n$ in tandem, where stage A_i consists of m_i parallel channels, each having the same constant service time s_i; Model II — the same as model I except that one of the stages A_j is a

single channel ($m_j = 1$) having variable service times which are always $\geqslant s_i/m_i$ for all $i \neq j$ (called 'dominant' service times). The arrival time sequences are supposed to be fixed but arbitrary.

Jackson (1962) has obtained the equilibrium state probabilities for a class of job-shop-like queuing models, in which the service rate at each centre is an essentially arbitrary function of the number of customers present there. Jackson (1962a) has also reviewed the work done on job-shop-like queuing models.

In addition to the work reviewed above, a number of papers on priority queues have also appeared. The more important of these are included in the bibliography of a paper by Hiller (1965).

Finally it may be remarked that the four volumes of the bibliography on Operations Research by Batchelor (1959, 1962, 1963, 1964) list most of the papers on queuing theory published till about 1963.

CHAPTER II

QUEUES WITH MIXED-MIXED-ERLANGIAN DISTRIBUTIONS

Queues with Erlangian arrival (or service) time distributions are too well known. A detailed analysis when arrival (or service) time distribution is Erlangian is found in Morse (1958), who has also considered the queuing system with Erlangian arrival and service time distributions. Kawamura (1961, 1962) has given some tables which in particular include the queuing system $E_l/E_k/1$. Gaver (1954), in an attempt to encompass a wide variety of service time distributions, introduced the concept of an infinite number of phases in the service channel when incoming units are not necessarily to traverse a fixed number of phases, as in the case of Erlang distribution. In fact, he introduced the probabilities, e_r, $r = 1,2,\ldots.,$ that a unit will have to traverse through r phases only. Later, Luchak (1956) considered a similar problem and preferred to fix a bound on r. He defined $p_n(t)$ as the probability that at time t, there are n phases to be served, including the one being served, and found the distribution function of $p_n(t)$. This had the obvious defect that we could know only the phase length distribution and not the queue length distribution. It was Jaiswal (1961) who studied the queuing system $1M-E_j/M/1$ and determined the system size distribution. Similarly Jain (1962) studied the queuing system $M/1M-E_j/1$ and determined the system size distribution.

Both Jaiswal (1961) and Jain (1962) have assumed the arrival (or service) parameter for all the j phases of the arrival (or service) channel to be the same. This obviously does not cover those applications in which the arrival (or service) parameter for all the j phases of the arrival (or service) channel are not same. Moreover, the purpose of introducing the probabilities c_r is

to accelerate or decelerate the input (or service) rate. The same purpose can also be achieved by the introduction of different arrival (or service) parameters for the different phases of the arrival (or service) channels.

In this chapter, we study the queuing systems in which both c_r and different arrival (or service) parameters for the various phases of the arrival (or service) channel are introduced. In short, section I is devoted to the study of the queuing system M-M-E_j/M/1 and section II to the study of the queuing system M/M-M-E_j/1.

SECTION I: The Queuing System M-M-E_j/M/1

The following queuing problem is considered in this section:

Units arrive at a single server service station where the service is offered according to the negative exponential distribution with mean $1/\mu$. The arrival channel consists of a number, j, of phases placed in the reverse order. The unit requiring service chooses (or is allotted) a number, r, of phases, $r = 1,2,\ldots,j$, with probability c_r, in each one of which it is first served in order before joining the queue or the service channel. The time a unit stays in the rth phase is distributed according to the negative exponential distribution with mean $1/\lambda_r$. At any time there is and can be only one unit in the whole arrival channel consisting of the j phases. Actually we are assuming the existence of a reservoir attached before the jth phase of the arrival channel which emits a unit as soon as the arrival channel is free. The unit joins the queue or service only from the first phase. The queue discipline is first come, first served. The system (queue + service) can accommodate only N units, so that when a unit comes from the arrival channel and finds N units already in the system, it goes away and is thus lost to the system.

Continuity Equations and Their Solution

Let us introduce $p(n,r)$ as the steady state probability that there are n units in the system, the unit in the arrival channel being in the rth phase. The continuity equations connecting the $p(n,r)$ in the steady state case are:

$$-\lambda_r p(0,r) + \lambda_{r+1}(1-\delta_{rj})p(0,r+1) + \mu\, p(1,r) = 0 \qquad (1)$$

$$-(\lambda_r + \mu)p(n,r) + \lambda_{r+1}(1-\delta_{rj})p(n,r+1) + \lambda_1 c_r\, p(n-1,1) + \mu\, p(n+1,r) = 0 \qquad (2)$$

$$(n = 1,2,\ldots,N-1)$$

$$-(\lambda_r + \mu)p(N,r) + \lambda_{r+1}(1-\delta_{rj})p(N,r+1) + \lambda_1 c_r\, p(N-1,1) + \lambda_1 c_r\, p(N,1) = 0 \qquad (3)$$

where δ_{rj} is the Kronecker delta and the equations (1), (2) and (3) are valid for $r = 1,2,\ldots,j$.

Also we have the following equation stating the condition of normality

$$\sum_{n=0}^{N}\sum_{r=1}^{j} p(n,r) = 1. \qquad (4)$$

Our problem is to solve equations (1) through (4) for the $(N+1)j$ probabilities involved. Equations (1) through (3) are $(N+1)j$ linear homogeneous equations in as many unknowns and thus a non-trivial solution will exist provided the determinant formed by the coefficients of these probabilities is zero. But equation (4) states that a non-trivial solution exists and hence the determinant formed by the coefficients is zero. Herebelow, we solve this system of equations by using the technique of generating functions.

Let us introduce the generating functions

$$F_r(x) = \sum_{n=0}^{N} p(n,r)\, x^n \qquad (5)$$

Multiplying equations (1), (2) and (3) by appropriate powers of x, adding and using (5), we get

$$\left[-(\lambda_r+\mu)x+\mu\right]F_r(x) + x\,\lambda_{r+1}(1-\delta_{rj})F_{r+1}(x) + \mu(x-1)p(0,r)$$
$$+\lambda_1 c_r\, x\left[x\,F_1(x) + p(N,1)x^N(1-x)\right] = 0 \qquad (6)$$

For $r = 1,2,\ldots,j-1$, we have from (6)

$$F_{r+1}(x) = A_r\,F_r(x) + B_r \qquad (7)$$

where

$$A_r = \frac{1}{x\,\lambda_{r+1}}\left[(\lambda_r+\mu)x-\mu\right]$$

$$B_r = \frac{1}{x\,\lambda_{r+1}}\Big[\mu(1-x)\,p(0,r) - \lambda_1 c_r\,x^2\,F_1(x) -$$
$$\lambda_1 c_r\,x^{N+1}(1-x)\,p(N,1)\Big],$$

On using equation (7) repeatedly on the right hand side, we get

$$F_{r+1}(x) = \prod_{i=1}^{r}A_i\,F_1(x) - \lambda_1\,x\,F_1(x)\sum_{i=1}^{r}\left[\frac{c_i}{\lambda_{i+1}}\left(\prod_{s=i+1}^{r}A_s\right)\right] +$$
$$\sum_{i=1}^{r}\left[B_i'\left(\prod_{s=i+1}^{r}A_s\right)\right] \qquad (8)$$

where

$$B_i' = B_i + \frac{\lambda_1 c_i\,x\,F_1(x)}{\lambda_{i+1}}.$$

Putting $r = j-1$ in (8) and $r = j$ in (6) and equating the two values thus obtained of $F_j(x)$, we obtain

$$F_1(x) = \frac{\dfrac{x-1}{(\lambda_j+\mu)x-\mu}\left[\mu\,p(0,j) - x^{N+1}\lambda_1 c_j\,p(N,1)\right] - \sum_{i=1}^{j-1}\left[B_i'\left(\prod_{s=i+1}^{j-1}A_s\right)\right]}{\prod_{i=1}^{j-1}A_i - \lambda_1 x\sum_{i=1}^{j-1}\left[\dfrac{c_i}{\lambda_{i+1}}\left(\prod_{s=i+1}^{j-1}A_s\right)\right] - \dfrac{\lambda_1 x^2 c_j}{(\lambda_j+\mu)x-\mu}} \qquad (9)$$

$$= \frac{\left[\mu p(0,j) - x^{N+1}\, p(N,1)\, \lambda_1 c_j\right] x^{j-1} - \left[(\lambda_j+\mu)x - \mu\right] \sum_{i=1}^{j-1} B_i'' x^{i-1} \left(\prod_{s=i+1}^{j-1} A_s'\right)}{C \prod_{i=1}^{j} (x - x_i)} \tag{10}$$

where we have cancelled the common factor $(x - 1)$ from the numerator and the denominator, C is the coefficient of x^j in the denominator of (9), x_i are the other j zeros of the denominator of (9), and

$$B_i'' = \frac{x B_i'}{x - 1},$$

$$A_i' = x A_i.$$

Now the normalising equation (4) becomes

$$\sum_{r=1}^{j} F_r(1) = 1$$

which on putting $x = 1$ in (8) gives

$$F_1(1) = \left\{1 + \lambda_1 \sum_{r=1}^{j-1} \left[\frac{1}{\lambda_{r+1}}\left(1 - \sum_{i=1}^{r} c_i\right)\right]\right\}^{-1} \tag{11}$$

where $F_1(1)$ may be obtained from (10) by putting $x = 1$.

The generating functions $F_r(x)$, $r = 1,2,\ldots,j$, given by (8) and (10) involve $j + 1$ unknown probabilities, viz., $p(N,1)$ and $p(0,r)$ for $r = 1,2,\ldots,j$. The left hand side of (10) is a polynomial, so must therefore be the right hand side also. Thus the numerator must cancel the j zeros of the denominator. This analyticity condition gives rise to j linear homogeneous equations, involving the above mentioned $j + 1$ unknown probabilities. These j equations together with the normalising equation (11) are sufficient to determine the $j + 1$ unknown probabilities. Hence all the j generating functions are known and thus all the probabilities $p(n,r)$, it being the coefficient of x^n in the expansion of $F_r(x)$.

Evaluation of Queue Characteristics

Once the probabilities $p(n,r)$ are determined, we may evaluate the mean number of units in the system and the mean waiting time by using the following formulae:

$$\text{Mean number of units in the system} = \left(\frac{d}{dx}\sum_{r=1}^{j} P_r(x)\right)\Bigg|_{x=1} \tag{12}$$

$$\text{Mean waiting time} = \frac{1}{[P_1(1) - p(N,1)]\lambda}\left[\left(\frac{d}{dx}P_1(x)\right)\Bigg|_{x=1} - N\,p(N,1)\right] \tag{13}$$

Infinite Waiting Space

In case an infinite waiting space is available, i.e. there is no bound on N, then $p(N,1) \longrightarrow 0$ as $N \longrightarrow \infty$ so that the numerator in (10) involves only j unknown probabilities. Also we can now prove that one of the j x_i's in (10) lies outside $|x| < 1$, and since the infinite series on the left hand side of (10) has to converge inside the unit circle atleast, we get $j - 1$ equations involving j unknown probabilities due to analyticity condition. These $j - 1$ equations together with the normalizing equation (11) are now sufficient to determine the j unknown probabilities, $p(0,r)$, for $r = 1, 2, \ldots, j$.

The Queuing System $2M\text{-}E_2/M/1$

In this section we show how the procedure outlined above can be carried out for the queuing system $2M\text{-}E_2/M/1$. By giving various arbitrary values to the queue parameters, some numerical work has been done and graphs depicting the behaviour of (1) probability of no delay, (2) mean number of units in the system, and (3) mean waiting time have also been drawn.

Substituting

$$c_r = \delta_{r1}, \quad j = 2$$

in (10), we get

$$F_1(x) = \frac{1}{\lambda_1\lambda_2(x - x_1)(x - x_2)}\Big[\lambda_1\lambda_2 p(N,1)\, x^{N+2} - \mu\{(\lambda_2 + \mu)x - \mu\}\, p(0,1) - \mu\lambda_2 x\, p(0,2)\Big] \qquad (14)$$

where

$$x_1, x_2 = \frac{\mu}{2\lambda_1\lambda_2}\Big[(\lambda_1 + \lambda_2 + \mu) \pm \sqrt{(\lambda_1 + \lambda_2 + \mu)^2 - 4\lambda_1\lambda_2}\Big].$$

The three equations for the determination of $p(0,1)$, $p(0,2)$ and $p(N,1)$ are:

$$\lambda_1\lambda_2 x_1^{N+2}\, p(N,1) - \mu\{(\lambda_2 + \mu)x_1 - \mu\}\, p(0,1) - \mu\lambda_2 x_1\, p(0,2) = 0 \qquad (15)$$

$$\lambda_1\lambda_2 x_2^{N+2}\, p(N,1) - \mu\{(\lambda_2 + \mu)x_2 - \mu\}\, p(0,1) - \mu\lambda_2 x_2\, p(0,2) = 0 \qquad (16)$$

$$\lambda_1\, p(N,1) - \mu\, p(0,1) - \mu\, p(0,2) = \frac{\lambda_1\lambda_2(1-x_1)(1-x_2)}{\lambda_1 + \lambda_2} \qquad (17)$$

Equations (15), (16) and (17) give on solution

$$p(N,1) = \frac{\lambda_2(1 - x_1)(1 - x_2)(x_1 - x_2)}{(\lambda_1 + \lambda_2)\Big[(x_1 - x_2) + x_1x_2(x_1^{N+1} - x_2^{N+1}) - (x_1^{N+2} - x_2^{N+2})\Big]} \qquad (18)$$

$$p(0,1) = \frac{\lambda_1\lambda_2 x_1 x_2 (x_1^{N+1} - x_2^{N+1})\, p(N,1)}{\mu^2 (x_1 - x_2)} \qquad (19)$$

$$p(0,2) = \frac{\Big[\lambda_1(\lambda_2 + \mu)x_1x_2(x_2^{N+1} - x_1^{N+1}) + \mu\lambda_1(x_1^{N+2} - x_2^{N+2})\Big]\, p(N,1)}{\mu^2 (x_1 - x_2)} \qquad (20)$$

Also,

$$\text{Mean number of units in the system} = \left(\frac{d}{dx}\Big[F_1(x) + F_2(x)\Big]\right)\Bigg|_{x=1}$$

$$= \frac{1}{\lambda_2}\Big[(\lambda_1 + \lambda_2)\left(\frac{d}{dx}F_1(x)\right)\Big|_{x=1} + \mu\, p(0,2) + \frac{\lambda_1(\lambda_2 - \mu)}{\lambda_1 + \lambda_2} - \lambda_1\, p(N,1)\Big] \qquad (21)$$

where

$$\frac{d}{dx} F_1(x)\Big|_{x=1} = \frac{1}{\lambda_1\lambda_2(1-x_1)(1-x_2)}\Big[\lambda_1\lambda_2(N+2)p(N,1) - \mu(\lambda_2+\mu)p(0,1) - p(0,2)\mu\lambda_2\Big]$$

$$- \frac{2-(x_1+x_2)}{\lambda_1(1-x_1)^2\,(1-x_2)^2}\Big[\lambda_1\,p(N,1) - \mu\,p(0,1) - \mu p(0,2)\Big]$$

Graphs are drawn for the following queue characteristics:

(i) probability of no delay = $p(0,1) + p(0,2)$, equations (19) and (20)

(ii) Mean number of units in the system, equation (21)

(iii) Mean waiting time, equation (13),

against the ratio λ_1/λ_2 of the two input rates.

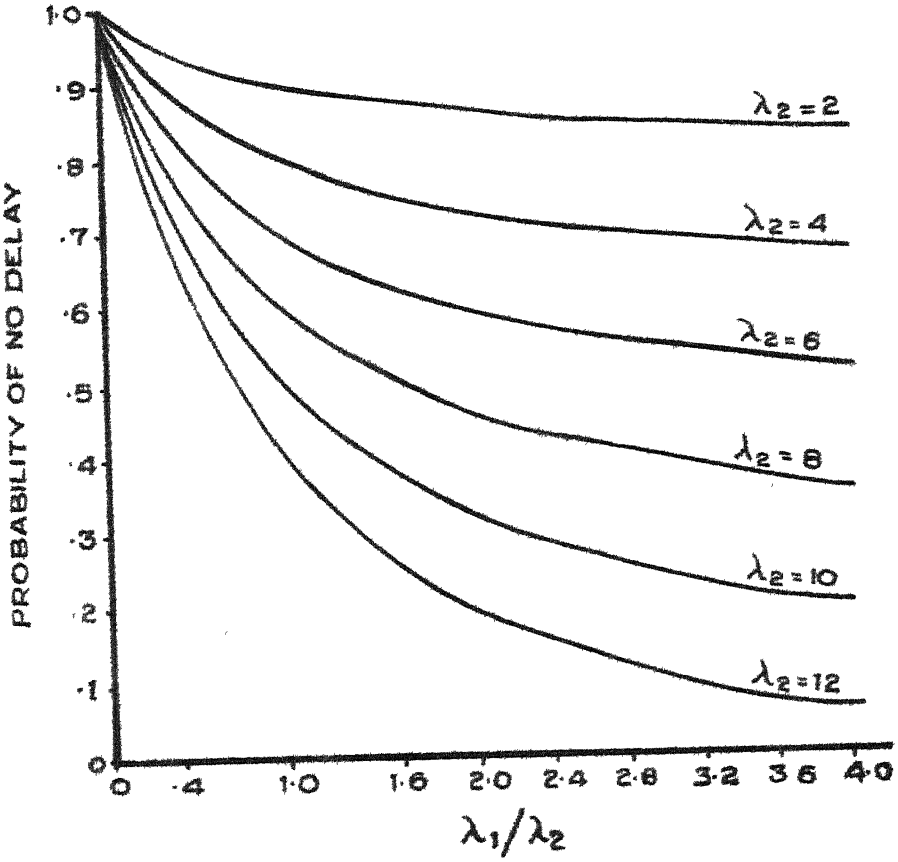
These graphs are drawn for $N = 20$, $\mu = 10$, and various values of λ_2 indicated in the graphs.

The results known earlier are just the points of intersections of these graphs with the line $\lambda_1/\lambda_2 = 1$.

The numerical calculations also show that:

(i) the value of $p(0,1) + p(0,2)$ does not change appreciably with N. The change is sometimes noticed only in the seventh or eighth decimal place.

(ii) the value of mean number of units in the system does not change appreciably with N till $\lambda_2 = 6$.


PROBABILITY OF NO DELAY
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
0
·4
1·0
1·6
2·0
2·4
2·8
3·2
3·6
4·0
λ_1/λ_2
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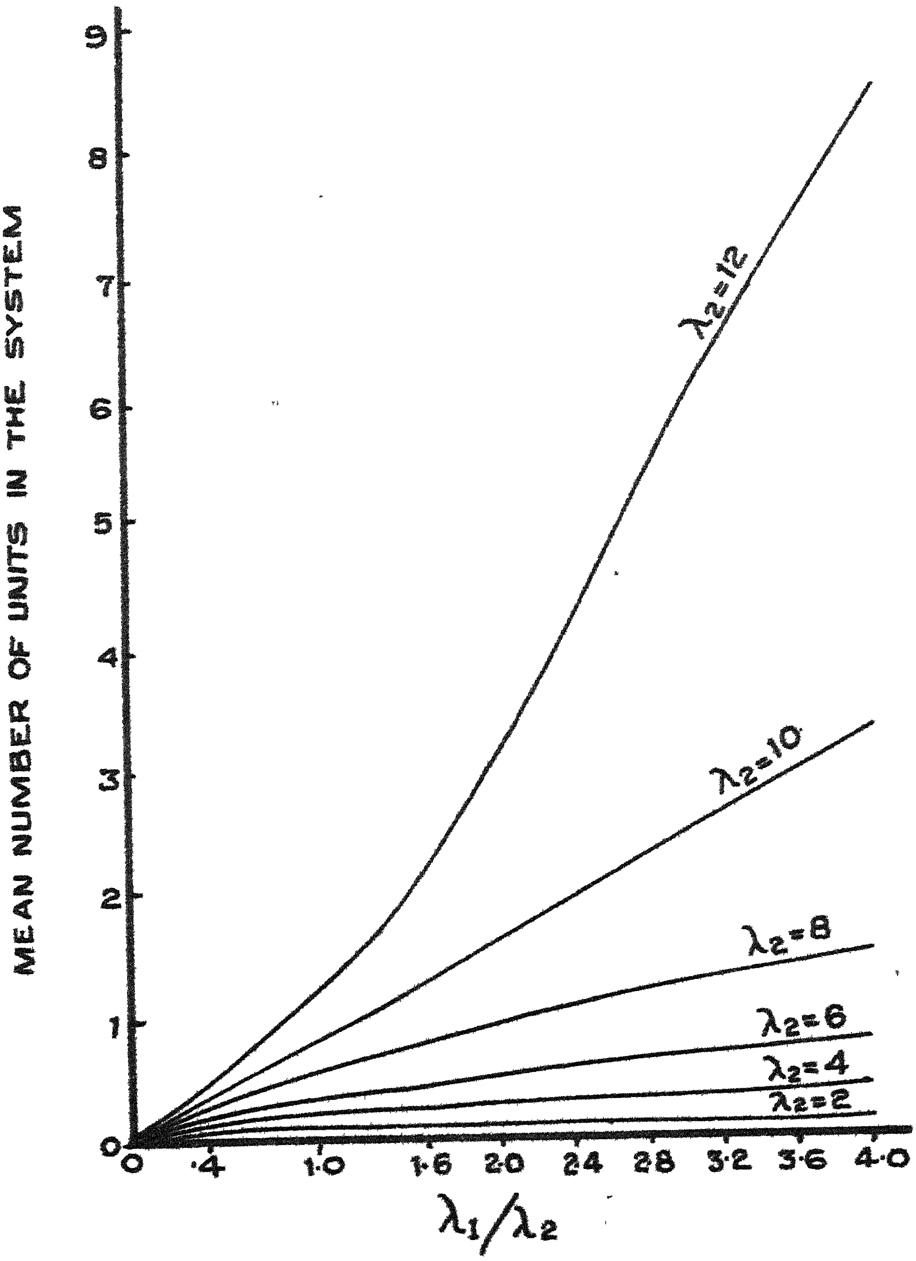
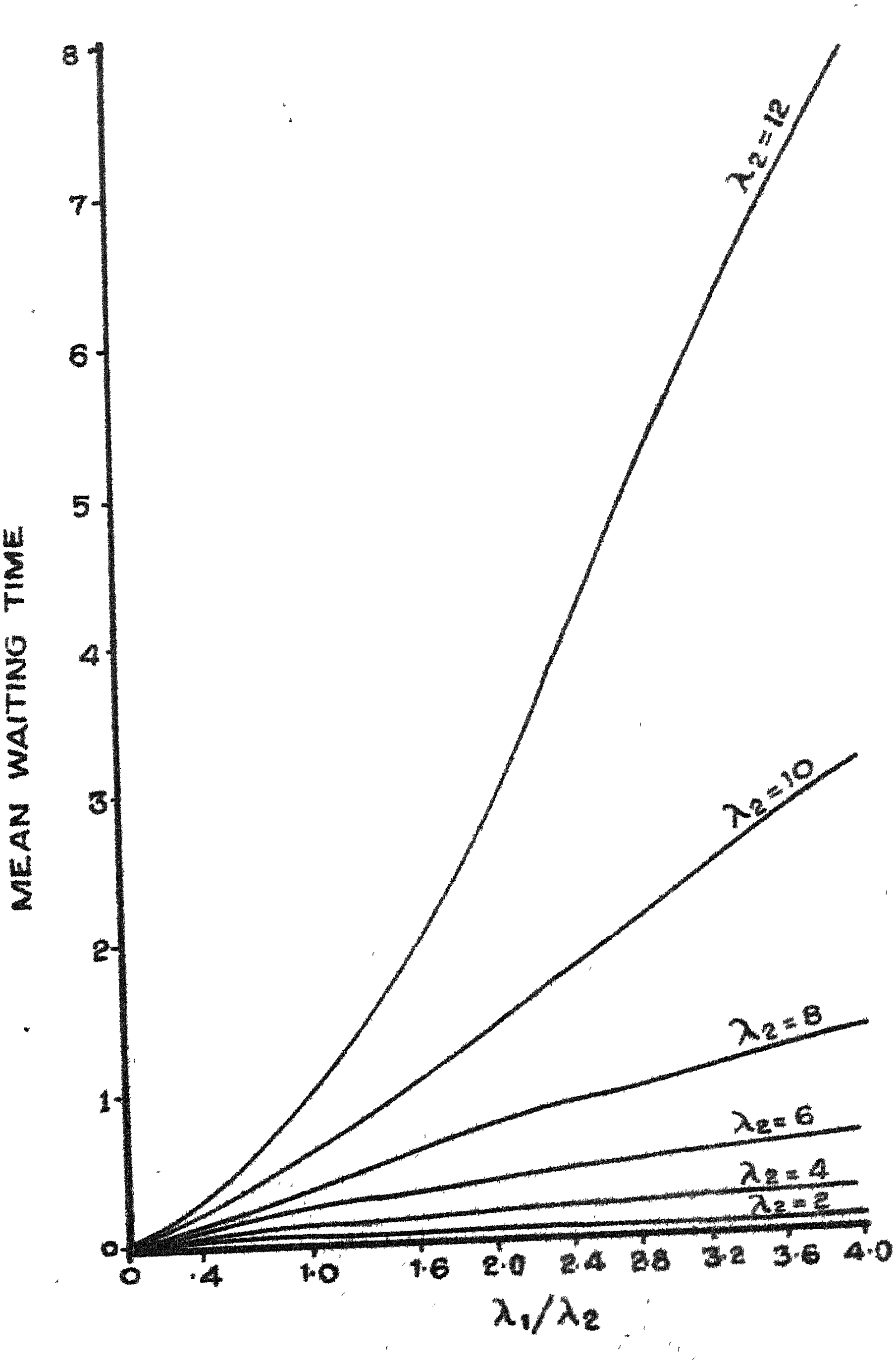
$\lambda_2 = 4$
$\lambda_2 = 6$
$\lambda_2 = 8$
$\lambda_2 = 10$
$\lambda_2 = 12$


MEAN NUMBER OF UNITS IN THE SYSTEM
9
8
7
6
5
4
3
2
1
0
0
·4
1·0
1·6
2·0
2·4
2·8
3·2
3·6
4·0
λ_1/λ_2
$\lambda_2=12$
$\lambda_2=10$
$\lambda_2=8$
$\lambda_2=6$
$\lambda_2=4$
$\lambda_2=2$


MEAN WAITING TIME
8
7
6
5
4
3
2
1
0
0
·4
1·0
1·6
2·0
2·4
2·8
3·2
3·6
4·0
λ_1/λ_2
$\lambda_2 = 12$
$\lambda_2 = 10$
$\lambda_2 = 8$
$\lambda_2 = 6$
$\lambda_2 = 4$
$\lambda_2 = 2$

SECTION II: The Queuing System $M/M\text{-}M\text{-}E_j/1$

The following problem is considered in this section:

Units arrive at a single server service facility according to a stationary Poisson stream with parameter λ. The service facility consists of a number, j, of phases. Each arriving unit chooses (or is allotted) a number, r, of phases, $r = 1,2,\ldots,j$, with probability c_r, in each one of which it is served before leaving the service facility. The service time distribution in the rth phase of the service facility is assumed to be negative exponential with parameter μ_r. At any time there is and can be only one unit present in the whole service channel consisting of the j phases. The incoming unit joins the queue or service channel according as the service channel is busy or not. The queue discipline is first come, first served. The system (queue + service) can accommodate only N units, so that when a unit arrives and finds N units already in the system, it goes away and is thus lost to the system.

Continuity Equations and Their Solution

Let us introduce $p(n,r)$ as the steady state probability that there are n units in the system the unit in the service channel being in the rth phase. Also let $p(0)$ denote the probability of there being no unit in the system. Thus our phase space consists of the point corresponding to $p(0)$ and the elements of the Cartesian product $J \times R$ where J and R are the finite sets $\{1, 2, \ldots, N\}$ and $\{1, 2, \ldots, j\}$ respectively. The continuity equations connecting the $p(n,r)$ in the steady state case are:

$$-(\lambda + \mu_r)\, p(n,r) + \mu_{r+1}(1 - \delta_{rj})\, p(n,r+1) + \lambda\, p(n-1,r) + \mu_1 c_r\, p(n+1,1) = 0 \qquad (22)$$

$$(n = 2,3,\ldots,N-1)$$

$$- (\lambda + \mu_r)\, p(1,r) + \mu_{r+1} (1 - \delta_{rj})\, p(1,r+1) + \lambda c_r\, p(0) +$$

$$\mu_1 c_r\, p(2,1) = 0 \tag{23}$$

$$- \mu_r p(N,r) + \mu_{r+1} (1 - \delta_{rj})\, p(N,r+1) + \lambda\, p(N-1,r) = 0 \tag{24}$$

$$- \lambda\, p(0) + \mu_1\, p(1,1) = 0 \tag{25}$$

where δ_{rj} is the Kronecker delta and equations (22) through (25) are valid for $r = 1,2,\ldots\ldots,j$.

Also we have the following equation stating the condition of normality

$$\sum_{n=1}^{N} \sum_{r=1}^{j} p(n,r) + p(0) = 1 \tag{26}$$

Our problem is to solve the equations (22) through (26) for the $Nj + 1$ probabilities involved. Equations (22) through (25) are $Nj + 1$ linear homogeneous equations in as many unknowns and thus a non-trivial solution will exist provided the determinant formed by the coefficients of these probabilities is zero. But equation (26) states that a non-trivial solution exists and hence the determinant formed by the coefficients is zero. Herebelow, we solve this system of equations by using the technique of generating functions.

Let us define the generating functions

$$F_r(x) = \sum_{n=1}^{N} p(n,r)\, x^n \tag{27}$$

Multiplying equations (22) through (24) by appropriate powers of x, adding and using (25) and (27), we get for $r = 1,2,\ldots,j$

$$\left[- (\lambda + \mu_r) + \lambda x \right] F_r(x) + \mu_{r+1} (1 - \delta_{rj}) F_{r+1}(x) + \frac{\mu_1 c_r}{x} F_1(x)$$

$$+ \lambda x^N p(N,r)\, (1-x) + \lambda c_r\, p(0)(x-1) = 0 \tag{28}$$

For $r = 1,2,\ldots,j-1$, we have from (28)

$$F_{r+1}(x) = A_r F_r(x) + B_r \tag{29}$$

where

$$A_r = \frac{1}{\mu_{r+1}}\left[\lambda(1-x) + \mu_r\right],$$

$$B_r = \frac{1}{\mu_{r+1}}\left[(x-1)\left\{\lambda x^N p(N,r) - \lambda c_r p(0)\right\} - \frac{\mu_1 c_r F_1(x)}{x}\right].$$

Using (29) repeatedly on the right hand side of (29), we obtain for $r = 1,2,\ldots,j-1$

$$F_{r+1}(x) = F_1(x)\prod_{i=1}^{r} A_i + \sum_{i=1}^{r}\left[\left(\prod_{s=i+1}^{r} A_s\right)\left\{B_i' - \frac{\mu_1 c_i F_1(x)}{x\,\mu_{i+1}}\right\}\right] \tag{30}$$

where

$$B_i' = B_i + \frac{\mu_1 c_i F_1(x)}{x\,\mu_{i+1}}.$$

Putting $r = j-1$ in (30) and $r = j$ in (28) and equating the two values of $F_j(x)$ thus obtained, we get

$$F_1(x) = \frac{x(1-x)\left[\lambda c_j p(0) - \lambda x^N p(N,j) + \{\lambda x - (\lambda+\mu_j)\}\sum_{i=1}^{j-1}\left(B_i'' \prod_{s=i+1}^{j-1} A_s\right)\right]}{\left[\lambda x - (\lambda+\mu_j)\right]\left[x\prod_{i=1}^{j-1}A_i - \mu_1\sum_{i=1}^{j-1}\frac{c_i}{\mu_{i+1}}\left(\prod_{s=i+1}^{j-1}A_s\right)\right] + \mu_1 c_j} \tag{31}$$

$$= \frac{x\left[\lambda c_j p(0) - \lambda x^N p(N,j) + \{\lambda x - (\lambda+\mu_j)\}\sum_{i=1}^{j-1}\left\{B_i'' \prod_{s=i+1}^{j-1} A_s\right\}\right]}{C\prod_{p=1}^{j}(x - x_p)} \tag{32}$$

where

$$B_i'' = B_i'/(x-1)$$

C is the coefficient of x^j in the denominator and $x_1, x_2, \ldots, x_j$ are the j

roots of the denominator different from unity which has been cancelled both from the numerator as also from the denominator.

Since now $F_1(x)$ is a polynomial in x, it is analytic. Therefore, the numerator on the right hand side of (32) must cancel the j zeros of the denominator, which gives rise to j homogeneous linear equtions involving j + 1 unknown probabilities.

Also the normalising equation (26) gives

$$F_1(1)\left[1+\mu_1\sum_{\gamma=1}^{j-1}\left(\frac{1-\sum_{i=1}^{\gamma}c_i}{\mu_{\gamma+1}}\right)\right]+p(0)=1 \qquad (33)$$

where $F_1(1)$ may be obtained from (32) by putting $x = 1$.

Thus the j equations due to the analyticity of the right hand side of (32) and equation (33) are sufficient to determine the j + 1 unknown probabilities involved in $F_1(x)$ given by (32). Once $F_1(x)$ is determined the other generating functions may be obtained from(3).

Infinite Waiting Space

In case an infinite waiting space is available, i.e. $N\longrightarrow\infty$, then $p(N,r)\longrightarrow 0$ for $r = 1,2,\ldots,j$ and $F_1(x)$ in (32) becomes an infinite series and should converge atleast in the region $|x|<1$. But we can easily prove by using Rouche's theorem that the j roots of the denominator of the right hand side of (32) all lie outside the region $|x|<1$. Also the numerator contains only one unknown probability, viz., p(0), which may now be calculated from the normalizing condition (33). Thus, we have

$$p(0)=\left[1+\frac{\lambda c_1}{c\prod_{p=1}(1-x_p)}\left\{1+\mu_1\sum_{\gamma=1}^{j-1}\left(\frac{1-\sum_{i=1}^{\gamma}c_i}{\mu_{\gamma+1}}\right)\right\}\right]^{-1} \qquad (34)$$

and therefore $F_1(x)$ and hence $F_r(x)$ for $r = 1,2,\ldots,j$ are completely determined in this case.

The Queuing System $M/2M\text{-}E_j/1$ (Infinite Queue Case)

Substituting

$$c_r = \delta_{rj}$$

in the analysis above, we get the results for the queuing system $M/2M\text{-}E_j/1$. Thus, if an infinite queue is allowed, we have

$$F_1(x) = \frac{\lambda x (1-x)\, p(0)}{\mu_1 + x\left[\lambda x - (\lambda + \mu_j)\right]\left(\prod_{i=1}^{j-1} A_i\right)} \tag{35}$$

$$F_{r+1}(x) = F_1(x) \prod_{i=1}^{r} A_i \tag{36}$$

$$p(0) = \left[1 + \frac{\lambda}{C' \prod_{p=1}^{j}(1-x_p')}\left\{1 + \mu_1 \sum_{r=1}^{j-1} \frac{1}{\mu_{r+1}}\right\}\right]^{-1} \tag{37}$$

where now x_p', $p = 1,2,\ldots,j$, are the j roots of the equation

$$\mu_1 + x\left[\lambda x - (\lambda + \mu_j)\right]\left(\prod_{i=1}^{j-1} A_i\right) = 0 \tag{38}$$

different from unity and C' is the coefficient of x^j in the expression on the left hand side of (38) after $(1-x)$ has been taken out as a common factor.

The Queuing System $M/2M\text{-}E_2/1$ (Finite Queue Case)

In particular, if we substitute

$$j = 2; \quad c_r = \delta_{rj}$$

in (32), we get

$$F_1(x) = \frac{x}{\lambda^2 (x-x_1')(x-x_2')} \Big[\lambda\mu_2 p(0) - \lambda\mu_2 x^N p(N,2) + \lambda x^N p(N,1) \cdot \{\lambda x - (\lambda + \mu_2)\}\Big] \qquad (39)$$

where

$$x_1', x_2' = \frac{1}{2\lambda}\Big[(\lambda+\mu_1+\mu_2) \pm \sqrt{(\lambda+\mu_1+\mu_2)^2 - 4\mu_1\mu_2}\Big].$$

The three equations for the determination of $p(N,1)$, $p(N,2)$ and $p(0)$ in this case are:

$$\big[(\lambda+\mu_2) - \lambda y_1\big]\lambda y_1^N p(N,1) + \lambda\mu_2 y_1^N p(N,2) - \lambda\mu_2 p(0) = 0 \qquad (40)$$

$$\big[(\lambda+\mu_2) - \lambda y_2\big]\lambda y_2^N p(N,1) + \lambda\mu_2 y_2^N p(N,2) - \lambda\mu_2 p(0) = 0 \qquad (41)$$

$$F_1(1) = \frac{\mu_2}{\mu_1+\mu_2} \qquad (42)$$

where

$$y_i = x_i', \quad i = 1,2.$$

Equations (40), (41) and (42) on solution give:

$$p(0) = \frac{\lambda(1-y_1)(1-y_2)\, y_1^N y_2^N (y_2 - y_1)}{(\mu_1+\mu_2)\big[y_1^N y_2^N (y_2-y_1) - (y_1^N - y_2^N) - (y_2^{N+1} - y_1^{N+1})\big]} \qquad (43)$$

$$p(N,1) = \frac{\mu_2 (y_2^N - y_1^N)\, p(0)}{\lambda y_1^N y_2^N (y_2 - y_1)} \qquad (44)$$

$$p(N,2) = \frac{\big[(\lambda+\mu_2)(y_1^N - y_2^N) + \lambda(y_2^{N+1} - y_1^{N+1})\big]\, p(0)}{\lambda y_1^N y_2^N (y_2 - y_1)} \qquad (45)$$

Graphs are drawn for the following queue characteristics:

(i) probability of no delay, i.e. $p(0)$, equation (43),

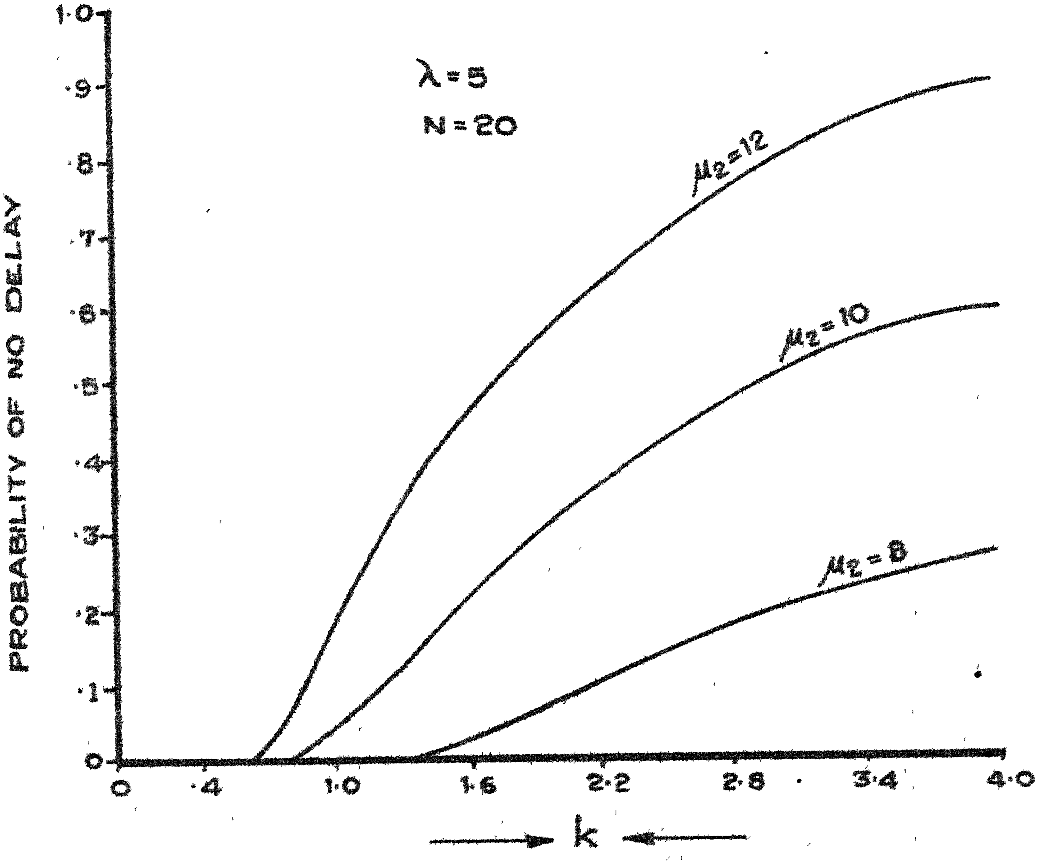
(ii) probability of loss, i.e. $p(N,1) + p(N,2)$, equations (44) and (45),

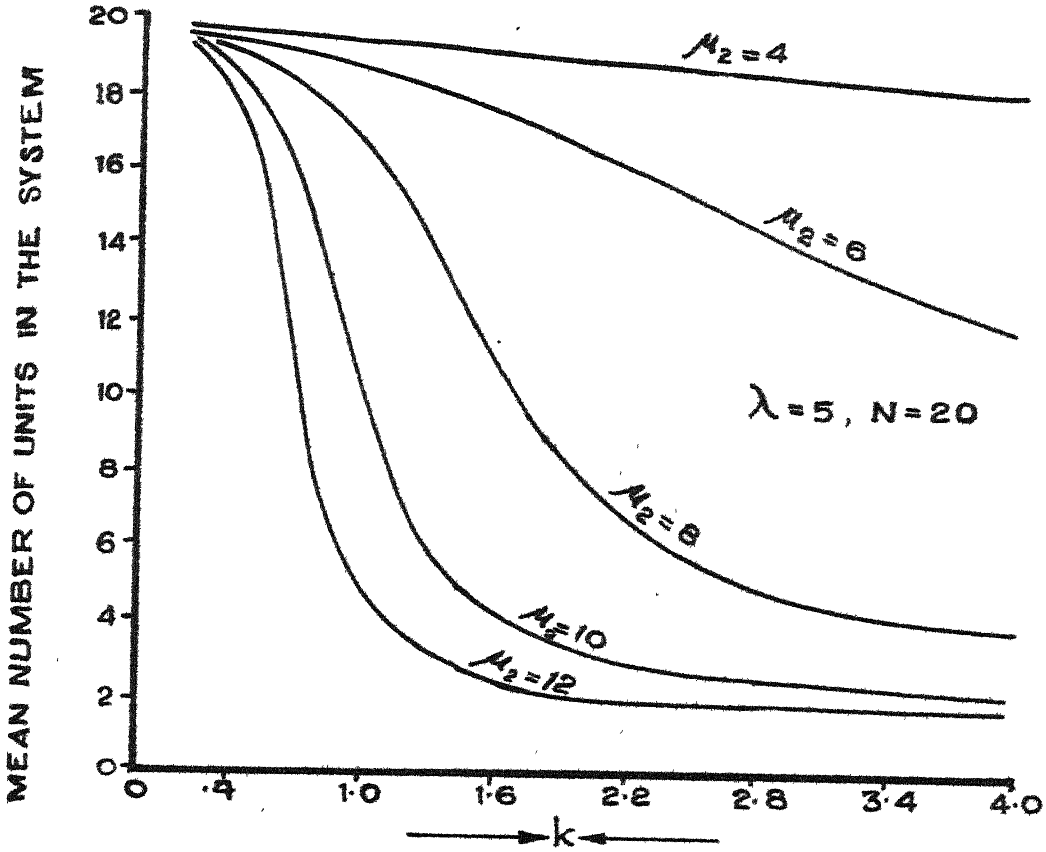
(iii) mean number, M, of units in the system by using the formula

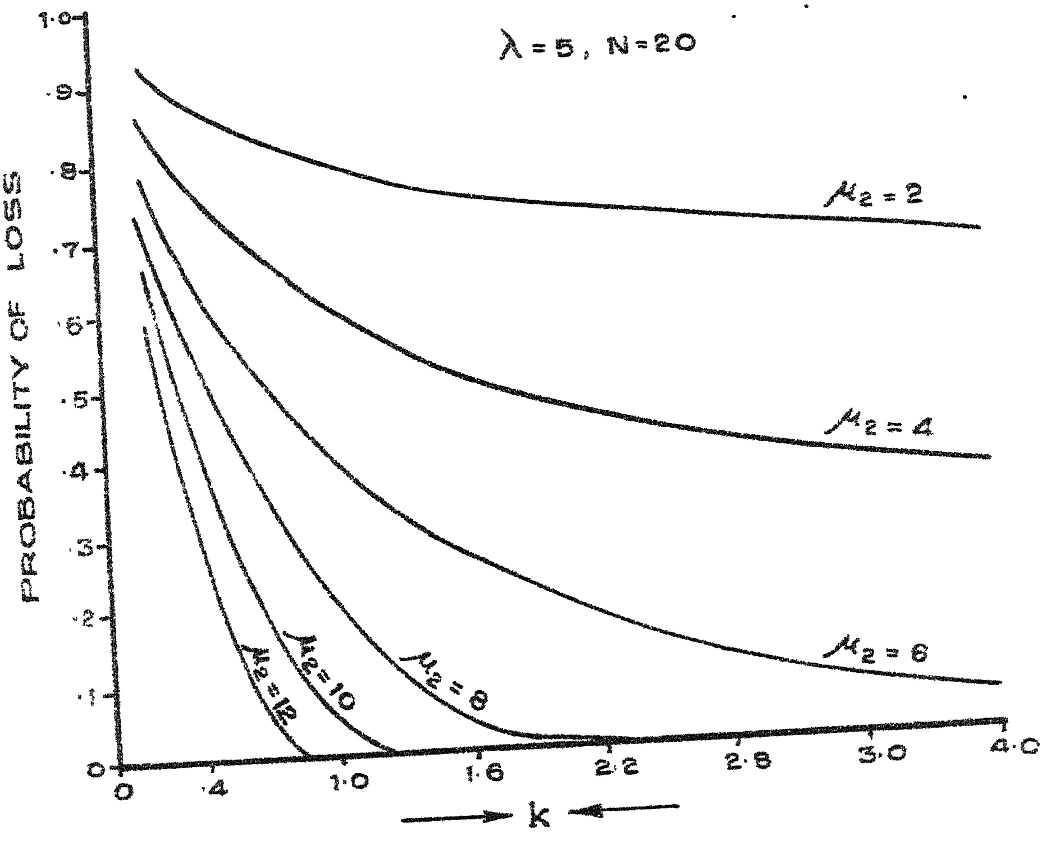
$$M = \left\{ \frac{d}{dx} \left[F_1(x) + F_2(x) \right] \right\} \Big|_{x=1}$$

against the ratio $k = \mu_1/\mu_2$ of the two mean service rates. The results known earlier are just the points of intersections of these graphs with the line $k = 1$. These graphs are drawn for $N = 20$, $\lambda = 5$, and various values of μ_2 indicated in the graphs.

The numerical calculations also show that the value of $p(0)$ does not change appreciably with N, whereas that of $p(N,1) + p(N,2)$ changes.


λ = 5
N = 20
$\mu_2 = 12$
$\mu_2 = 10$
$\mu_2 = 8$
PROBABILITY OF NO DELAY
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
0
·4
1·0
1·6
2·2
2·8
3·4
4·0
k


MEAN NUMBER OF UNITS IN THE SYSTEM
0
2
4
6
8
10
12
14
16
18
20
0
·4
1·0
1·6
2·2
2·8
3·4
4·0
k
$\mu_2 = 4$
$\mu_2 = 6$
$\mu_2 = 8$
$\mu_2 = 10$
$\mu_2 = 12$
$\lambda = 5$, $N = 20$


λ = 5, N = 20
PROBABILITY OF LOSS
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
$\mu_2 = 2$
$\mu_2 = 4$
$\mu_2 = 6$
$\mu_2 = 8$
$\mu_2 = 10$
$\mu_2 = 12$
0
·4
1·0
1·6
2·2
2·8
3·0
4·0
k

CHAPTER III

QUEUES WITH HYPER DISTRIBUTIONS

Morse (1958) considered the steady state case of the queuing systems $H\text{-}M_2/M/1$ and $M/H\text{-}M_2/1$ when an infinite queue is allowed. Gupta and Goyal (1964a, 1964b) generalised Morse's result, inasmuchas they considered the queuing systems $H\text{-}M_n/M/1$ and $M/H\text{-}M_n/1$ when only a finite queue is allowed. Also the mean arrival (or service) rates from (or in) the various branches taken by them was more general than that taken by Morse (1958). Jaiswal (1961) studied the transient case of the queuing system $1M\text{-}E_j/M/1$ when only a finite queue is allowed and Jain (1962) also studied similarly the steady state case of the queuing system $M/1M\text{-}E_j/1$. Later, however, the author (1965c, 1965a) studied the transient case of the queuing systems $H\text{-}1M\text{-}E_{nk}/M/1$ and $M/H\text{-}1M\text{-}E_{nk}/1$. Again, he (1965b) considered the transient case of the queuing system $M/H\text{-}G_n/1$. For the transient case, the Laplace transforms of the system size distributions could only be obtained.

This chapter is devoted to the study of these three queuing systems — $M/H\text{-}1M\text{-}E_{nk}/1$, $H\text{-}1M\text{-}E_{nk}/M/1$ and $M/H\text{-}G_n/1$, studied respectively in the sections I, II and III. The results for the queuing systems $H\text{-}M_n/M/1$ and $M/H\text{-}M_n/1$ are deduced as particular cases. In section III, we also deduce the system size distribution for the queuing system $M/H\text{-}D_n/1$ as a particular case.

SECTION I: The Queuing System $M/H\text{-}1M\text{-}E_{nk}/1$

The following problem is considered in this section:

Suppose that units requiring various services arrive at random at the mean rate of λ per unit time and are served in order of arrival at a service channel

consisting of n independent branches and each branch consisting of k phases with different rates of service in each branch only. The whole service channel is busy if a unit is present (being served) in any one of the nk phases and in case the service channel is busy the arriving units go on forming a queue, the maximum number of units allowed in the system (queue + service) being N. The moment the service channel disposes of the unit being served, the unit at the head of the queue, if there be any, enters at random any one of the n branches of the service channel. The fraction of time (on the average) it goes to the rth branch is σ_r, so that $\sum_{r=1}^{n} \sigma_r = 1$. The service time distribution in the rth branch is mixed-Erlangian of the first kind with the same mean μ_r in each of the phases of the branch. By mixed-Erlangian distribution, I mean that a unit which enters the rth branch, requires service in s phases, $s = 1,2,\ldots,k$, with probability d_s, so that $\sum_{s=1}^{k} d_s = 1$, where the service time distribution in each phase is exponential with mean μ_r and once a unit demands service in a certain number, s, of phases, it has to pass through each of the s phases, one after the other so that only the first (k-s) phases are not used. The phases in all the n branches, we assume are in reverse order, i.e. phase k is the first phase and phase 1 is the last phase. Any unit which enters the service channel can leave it only from the first phase of some branch.

Continuity Equations and Their Solution

Let $p(m,s,r,t)$ denote the probability that at time t there are m units in the system, the unit in service being in the sth phase of the rth branch. Also let $p(0,t)$ denote the probability of there being no unit in the system at time t. Assuming that the maximum number of units allowed in the system is N, and noting that $p(m,k+1,r,t) = 0$ for all m and r, we have the following continuity equations:

$$\frac{d}{dt} p(1,s,r,t) = -(\lambda + \mu_r)\, p(1,s,r,t) + \sigma_{\bar{r}} d_s \sum_{i=1}^{n} \mu_i\, p(2,1,i,t) + \mu_r\, p(1,s+1,r,t) + \lambda \sigma_{\bar{r}} d_s\, p(0,t) \qquad (1)$$

$$\frac{d}{dt} p(m,s,r,t) = -(\lambda + \mu_r)\, p(m,s,r,t) + \sigma_{\bar{r}} d_s \sum_{i=1}^{n} \mu_i\, p(m+1,1,i,t) + \mu_r\, p(m,s+1,r,t) + \lambda\, p(m-1,s,r,t) \qquad (2)$$

$$(m = 2,3,\ldots,N-1)$$

$$\frac{d}{dt} p(N,s,r,t) = -\mu_r\, p(N,s,r,t) + \mu_r\, p(N,s+1,r,t) + \lambda\, p(N-1,s,r,t) \qquad (3)$$

$$\frac{d}{dt} p(0,t) = -\lambda\, p(0,t) + \sum_{i=1}^{n} \mu_i\, p(1,1,i,t) \qquad (4)$$

where the equations (1), (2), (3) and (4) are valied for $s = 1,2,\ldots,k$ and $r = 1,2,\ldots,n$.

The system of differential-difference equations (1) through (4) represent $nNk + 1$ independent equations in as many unknowns. Below we solve this system analytically by using the technique of generating functions

Let us introduce the generating functions

$$F(m,z,r,t) = \sum_{s=1}^{k} p(m,s,r,t)\, z^s \qquad (5)$$

$$G(w,z,r,t) = \sum_{m=1}^{N} F(m,z,r,t)\, w^m \qquad (6)$$

$$H(w,1,r,t) = \sum_{m=1}^{N} p(m,1,r,t)\, w^m \qquad (7)$$

and

$$D(z) = \sum_{s=1}^{k} d_s\, z^s \qquad (8)$$

Multiplying equations (1), (2) and (3) by appropriate powers of z and w, using (4), (5), (6), (7) and (8) and applying the Laplace transformation

$$\bar{f}(u) = \int_0^\infty e^{-ut} f(t)\, dt \tag{9}$$

we get

$$\left[u + \lambda(1-w) + \mu_r(1 - 1/z)\right] \bar{G}(w,z,r,u) = \lambda w^N (1-w)\, \bar{P}(N,z,r,u) -$$

$$\sum_{i=1}^{n}\left[\left\{\delta_{ir} - \sigma_r D(z)/w\right\}\mu_i \bar{H}(w,1,i,u)\right] + D(z)\, w^p \sigma_r -$$

$$D(z)\, \sigma_r\, \bar{p}(0,u)\left[u + \lambda(1-w)\right] \tag{10}$$

where $p(>0)$ is the initial number of units with which the system starts at time $t = 0$.

Let us now set

$$z = \frac{\mu_r}{u + \lambda(1-w) + \mu_r} = X_r \quad \text{(say)}$$

in (10), then we get

$$\sum_{i=1}^{n}\left[\left\{\delta_{ir} - \sigma_r D(X_r)/w\right\}\mu_i \bar{H}(w,1,i,u)\right] = P_r - Q_r \tag{11}$$

where

$$P_r = \lambda w^N (1-w) \sum_{s=1}^{k} \bar{p}(N,s,r,u)\, X_r^{\,s}$$

and

$$Q_r = D(X_r)\, \sigma_r\, \bar{p}(0,u)\left[u + \lambda(1-w)\right] - D(X_r)\, \sigma_r\, w^p.$$

Now (11) can be written as the matrix equation

$$\underline{A}\underline{H} = \underline{B} \tag{12}$$

where $\underline{A}$ is the matrix given by

$$\underset{\sim}{A}' = \|a_{ij}\|$$

such that

$$a_{ij} = -D(x_i)\,\sigma_i\,\mu_j/w \qquad (i \neq j)$$

$$a_{ii} = \mu_i - D(X_i)\,\sigma_i\,\mu_i/w,$$

and $\underset{\sim}{H}$ and $\underset{\sim}{R}$ are column matrices

$$\left[\bar{H}(w,1,1,u),\ \bar{H}(w,1,2,u),\ldots\ldots,\bar{H}(w,1,n,u)\right]$$

and

$$\left[P_1 - Q_1,\ P_2 - Q_2,\ldots\ldots\ldots\ldots,\ P_n - Q_n\right]$$

respectively.

We observe that $\underset{\sim}{A}^{-1}$ is given by

$$\underset{\sim}{A}^{-1} = \|a'_{ij}\|$$

such that

$$a'_{ij} = \frac{D(X_i)\qquad{}_i}{\mu_i\,A(w)} \qquad (i \neq j)$$

$$a'_{ii} = \frac{A(w) + D(X_i)\sigma_i}{\mu_i\,A(w)}$$

where

$$A(w) = w - \sum_{j=1}^{n} D(X_j)\,\sigma_j\,.$$

Using this value of $\underset{\sim}{A}^{-1}$ in (12), we get

$$\bar{H}(w,1,r,u) = \frac{P_r - Q_r}{\mu_r} + \frac{D(X_r)\,\sigma_r \sum_{j=1}^{n}(P_j - Q_j)}{\mu_r\,A(w)} \qquad (13)$$

We now observe that the denominator of $\bar{H}(w,1,r,u)$ is a polynomial of order $nk + 1$. Also the unknown probabilities on the right hand side of (13) are $nk + 1$, viz. $\bar{p}(N,s,r,u)$ for $s = 1,2,\ldots\ldots,k$ and $r = 1,2,\ldots\ldots,n$ and $\bar{p}(0,u)$. Now $\bar{H}(w,1,r,u)$ being a polynomial is analytic, so must also be the right hand side of (13). In order that the right hand side be analytic, the numerator must cancel the $nk + 1$ zeros of the denominator which gives rise to $nk + 1$ equations in $nk + 1$ unknowns mentioned above. We can solve these $nk + 1$ equations for the unknowns and hence $\bar{H}(w,1,r,u)$ is completely known.

Substituting the values of $\bar{H}(w,1,r,u)$ and the unknowns determined above in (10), we determine $\bar{G}(w,z,r,u)$.

Let us define another function

$$\bar{G}(w,1,u) = \sum_{r=1}^{n} \bar{G}(w,1,r,u)$$

Substituting the values of $\bar{G}(w,1,r,u)$, we get

$$\bar{G}(w,1,u) = \frac{1}{u + \lambda(1-w)} \Bigg[w^{N} (1-w)\lambda \sum_{r=1}^{n} \sum_{s=1}^{k} \bar{p}(N,s,r,u) + w^{p} - \bar{p}(0,u) \left[u + \lambda(1-w) \right] - \sum_{r=1}^{n} \sum_{i=1}^{n} \left\{ \left(\delta_{ir} - \frac{\sigma_r \left\{ 1 - D(X_r) \right\}}{A(w)} \right) (P_i - Q_i) \right\} \Bigg]. \tag{14}$$

This is the function which represents the Laplace transform of the system-size distribution from which the queue characteristics can be calculated. For example,

Laplace transform of the mean number of units in the system

$$= \frac{d}{dw} \bar{G}(w,1,u) \Big|_{w=1}$$

$$= \frac{1}{u}\Bigg[\overline{G}(w,1,u) - \lambda\sum_{r=1}^{n}\sum_{s=1}^{k}\overline{p}(N,s,r,u) + p + \lambda\overline{p}(0,u) -$$

$$\sum_{r=1}^{n}\sum_{i=1}^{n}\left\{\left(\delta_{ir} - \frac{\{1 - D(Y_r)\}\sigma_r}{1 - \sum_{j=1}^{n}D(Y_j)\sigma_j}\right)\Big(\lambda D(Y_i)\,\sigma_i\overline{p}(0,u) + L_i\,\sigma_i +\right.$$

$$\left. pD(Y_i)\,\sigma_i - \lambda\sum_{s=1}^{k}\overline{p}(N,s,r,u)\;Y_i^{\,s} - uL_i\,\sigma_i\,\overline{p}(0,u)\Big)\right\} -$$

$$\frac{\{1 - u\overline{p}(0,u)\}\{\sum_{r=1}^{n}L_r\,\sigma_r\}\{\sum_{j=1}^{n}D(Y_j)\,\sigma_j\}}{1 - \sum_{j=1}^{n}D(Y_j)\,\sigma_j} -$$

$$\frac{\{1 - u\overline{p}(0,u)\}\{1 - \sum_{j=1}^{n}L_j\,\sigma_j\}\{\sum_{j=1}^{n}D(Y_j)\,\sigma_j\}\{\sum_{r=1}^{n}\sigma_r(1 - D(Y_r))\}}{\{1 - \sum_{j=1}^{n}D(Y_j)\sigma_j\}^2} \qquad (1$$

where

$$Y_r = \frac{\mu_r}{u + \mu_r}$$

and

$$L_r = \frac{\lambda\,(\sum_{s=1}^{k}s d_s\,Y_r^{\,s})}{u + \mu_r}\,.$$

Steady State Solution

Applying the well-known Abel's corollary

$$\lim_{u\to 0+} u\,\overline{f}(u) = \lim_{t\to\infty} f(t) = f \quad \text{(say)} \qquad (16)$$

to (14), we get the steady state solution of the problem under study. The result is:

$$G(w,1) = \frac{1}{\lambda(1-w)} \Big[w^N (1-w) \sum_{r=1}^{n} \sum_{s=1}^{k} p(N,s,r) - p(0)\, \lambda\, (1-w) -$$

$$\sum_{r=1}^{n} \sum_{i=1}^{n} \Big\{ \Big(\delta_{ir} - \frac{\sigma_r \{1 - D(X_r)\}}{w - \sum_{r=1}^{n} D(X_r)\, \sigma_r} \Big) (P_i - Q_i) \Big\} \Big] \tag{17}$$

where

$$P_r = \lambda w^N (1-w) \sum_{s=1}^{k} p(N,s,r)\, X_r^{\,s}$$

$$Q_r = D(X_r)\, \sigma_r\, p(0)\, \lambda\, (1-w)$$

$$X_r = \frac{\mu_r}{\lambda(1-w) + \mu_r}.$$

Infinite Waiting Space Case

In case an infinite waiting space is allowed, $\overline{p}(N,s,r,u) \longrightarrow 0$ as $N \longrightarrow \infty$. Thus there remains only one unknown, viz. $\overline{p}(0,u)$ on the right hand side of (13) and there is an infinite series on the left hand side. This infinite series must converge inside the unit circle at least. Also we can prove by applying Rouche's Theorem that there is only one zero of the denominator lying in the region determined by $|w| < 1$. Thus the unknown, $\overline{p}(0,u)$ can be calculate from the analyticity condition. If $w = w_1$, is the root of the denominator lying inside the unit circle, then

$$\overline{p}(0,u) = \frac{w_1^{\,p}}{u + \lambda(1-w)} \tag{18}$$

Making $\overline{p}(N,s,r,u) \longrightarrow 0$ and substituting the above value of $\overline{p}(0,u)$ from (18) above in (14) and (15) we get the Laplace transform of the system-

size function and the Laplace transform of the mean number of units in the case of infinite waiting space.

In particular, if we make these substitutions in (17), then we have in the <u>steady state infinite queue case</u>

$$G(w,1) = w\, p(0) \left[\frac{\sum_{j=1}^{n} D(X_j)\,\sigma_j - 1}{w - \sum_{j=1}^{n} D(X_j)\sigma_j} \right]. \tag{19}$$

The unknown $p(0)$ is now to be determined by the normalizing equation

$$G(1,1) + p(0) = 1 \tag{20}$$

which gives

$$p(0) = 1 - \rho \tag{21}$$

where

$$\rho = \lambda \left\{ \sum_{s=1}^{k} (s\, d_s) \right\} \left\{ \sum_{r=1}^{n} (\sigma_r / \mu_r) \right\},$$

the same as in the case of Poisson input and exponential service time distribution. Thus the number, n, of branches and the number, s, of phases does not affect the value of $p(0)$ in this case.

<u>The Queuing System M/H-M_n/1</u>

Substituting

$$d_s = \delta_{s1}$$

in (14) and writing $\overline{G}(w,u)$ for $\overline{G}(w,1,u)$, we get

$$\bar{G}(w,u) = \frac{1}{u + \lambda(1-w)}\Big[w^N (1-w) \sum_{r=1}^{n} \bar{p}(N,r,u) + w^p - \bar{p}(0,u)\left\{u + \lambda(1-w)\right\}$$

$$- \sum_{i=1}^{n}\sum_{r=1}^{n}\left\{\left(\delta_{ir} - \frac{\sigma_r(1-X_r)}{w - \sum_{j=1}^{n}\sigma_j X_j}\right)(P_i - Q_i)\right\}\Big] \tag{22}$$

where

$$P_r = \lambda w^N (1-w)\, \bar{p}(N,r,u)\, X_r$$

and

$$Q_r = X_r\, \sigma_r\, \bar{p}(0,u)\left[u + \lambda(1-w)\right] - X_r \sigma_r w^p.$$

The result contained in (22) represents the generalization of the result obtained earlier by Gupta and Goyal (1964a) in the sense that here we have obtained the Laplace transform of the time-dependant solution of the same problem.

In the steady state case, the case discussed by Gupta and Goyal (1964a), we get the following results:

$$G(w,r) = \lambda(1-w)\left[\frac{p(0)\sigma_r}{A_r} - \frac{w^N p(N,r)}{A_r} - \frac{\sigma_r}{A_r\left\{w + \sum_{j=1}^{n}\frac{\sigma_j \mu_j}{A_j}\right\}}\left\{p(0)\sum_{j=1}^{n}\frac{\sigma_j \mu_j}{A_j} - w^N \sum_{j=1}^{n}\frac{\mu_j\, p(N,j)}{A_j}\right\}\right] \tag{23}$$

where

$$A_r = \lambda(w-1) - \mu_r.$$

Thus we have expressed $G(w,r)$, $r = 1,2,\ldots,n$, in terms of $n+1$ unknowns $p(0)$ and $p(N,r)$. If we determine these $n+1$ unknowns then all the $nN+1$

probabilities $p(0)$ and $p(m,r)$ can be determined by comparing the different powers of w in $G(w,r)$. These $n+1$ unknowns are determined by the analyticity conditions of $G(w,r)$ and the normalizing equation.

We observe that the denominators of all the $G(w,r)$ are the same, viz. $\prod_{r=1}^{n} A_r + \lambda \sum_{r=1}^{n} \sigma_r \left(\prod_{\substack{i=1 \\ i \neq r}}^{n} A_i \right)$, i.e., a polynomial of degree n. Thus in order that $G(w,r)$ be all analytic, the numerators of $G(w,r)$ must cancel the zeros of the denominator, i.e. solutions of

$$\prod_{r=1}^{n} A_r + \lambda \sum_{r=1}^{n} \sigma_r \left(\prod_{\substack{i=1 \\ i \neq r}}^{n} A_i \right) = 0.$$

Since the denominator is a polynomial of order n, there are n zeros of the denominator. Taking any one of the n zeros, say $w = w_r$, we find that the numerators of all the $G(w,r)$ cancel this particular zero if the equation

$$p(0) \sum_{j=1}^{n} \sigma_j \mu_j \left(\prod_{\substack{i=1 \\ i \neq j}}^{n} A_{ir} \right) - w_r^N \sum_{j=1}^{n} \mu_j p(N,j) \left(\prod_{\substack{i=1 \\ i \neq j}}^{n} A_{ir} \right) = 0 \qquad (24)$$

is satisfied. Thus corresponding to the n zeros we get n equations of the type (24), where now

$$A_{ir} = \lambda (w_r - 1) - \mu_i$$

and w_r are the n zeros of the denominator.

Also,

$$G(w) = \sum_{r=1}^{n} G(w,r)$$

$$= \frac{-\lambda(1-w)}{\left[w + \sum_{r=1}^{n} \left(\frac{\sigma_r \mu_r}{A_r} \right)\right]} \left\{ p(0) \left[\sum_{r=1}^{n} \frac{\sigma_r \mu_r}{A_r} \right] \left[\sum_{j=1}^{n} \frac{\sigma_j}{A_j} \right] - w^N \left[\sum_{r=1}^{n} \frac{\mu_r p(N,r)}{A_r} \right] \left[\sum_{j=1}^{n} \frac{\sigma_j}{A_j} \right] \right\} + \qquad (25)$$

$$\lambda(1-w) \left\{ p(0) \sum_{j=1}^{n} \frac{\sigma_j}{A_j} - w^N \sum_{j=1}^{n} \frac{p(N,j)}{A_j} \right\}.$$

Now the normalising equation is

$$G(1) + p(0) = 1,$$

or

$$\frac{\lambda \sum_{j=1}^{n} \frac{\sigma_j}{\mu_j} \left[p(0) - \quad p(N,r) \right]}{1 - \lambda \sum_{j=1}^{n} \frac{\sigma_j}{\mu_j}} + \quad p(0) \quad = 1 \tag{26}$$

Thus the n equations represented by (24) together with (26) are sufficient to determine the n + 1 unknowns involved in G(w). Hence, in principle at least, we have determined all the nN + 1 probabilities.

G(w) given in (25) is the required function, representing the system-size distribution from which the various queue parameters can be calculated. For example,

Mean number of Units in the System = $\frac{d}{dw} G(w) \Big|_{w=1}$

$$= \lambda \left[p(0) \sum_{r=1}^{n} \frac{\sigma_r}{\lambda_r} - \sum_{r=1}^{n} \frac{p(N,r)}{\mu_r} \right] + \lambda^2 \left[p(0) - \sum_{r=1}^{n} p(N,r) \right] \cdot$$

$$\frac{\sum_{r=1}^{n} \frac{\sigma_r}{\mu_r^2}}{(1-\rho)^2} + \frac{\rho \left[p(0)\rho - N \sum_{r=1}^{n} p(N,r) + \lambda \sum_{r=1}^{n} \frac{p(N,r)}{\mu_r} \right]}{1-\rho} \tag{27}$$

where

$$\rho = \lambda \sum_{r=1}^{n} \frac{\sigma_r}{\mu_r} .$$

Subtracting, we get

$$2iq \sum_{r=1}^{n} \frac{\sigma_r}{(\mu_r/\lambda - p)^2 + q^2} = 0,$$

giving

$$q = 0.$$

Hence the roots are all real. Now we prove that no root lies inside $|w| < 1$.

Without any loss of generality, we can assume that

$$\mu_1 < \mu_2 < \mu_3 < \dots\dots\dots < \mu_n,$$

Let

$$f(x) = 1 + \sum_{r=1}^{n} \frac{\sigma_r}{x - \mu_r/\lambda} ;$$

then for $\epsilon > 0$,

$$\lim_{\epsilon \to 0} f(\mu_1/\lambda + \epsilon) \longrightarrow + \infty$$

and

$$\lim_{\epsilon \to 0} f(\mu_2/\lambda - \epsilon) \longrightarrow - \infty .$$

The function $f(x)$ is continuous between these values and hence must vanish at some point between $x = \mu_1/\lambda$ and μ_2/λ. Hence w has at least one value between $1 + \mu_1/\lambda$ and $1 + \mu_2/\lambda$. Similarly, it can be shown that w has at least one value between $1 + \mu_2/\lambda$ and $1 + \mu_3/\lambda$ and so on, the last one being between $1 + \mu_{n-1}/\lambda$ and $1 + \mu_n/\lambda$. Thus $n - 1$ zeros at least are greater than unity. Now,

$$\lim_{\epsilon \to 0} f(\mu_1/\lambda - \epsilon) \longrightarrow - \infty \quad \text{and} \quad f(0) > 0,$$

so one root lies between 1 and $1 + \mu_1/\lambda$. Since now there are in all, n roots, therefore we can say that in each of the intervals considered above, lies only one root. Hence, all the n roots lie outside the unit circle.

Thus, letting $p(N,r) \rightarrow 0$, we get from (26)

$$p(0) = 1 - \lambda \sum_{r=1}^{n} \frac{\sigma_r}{\mu_r} = 1 - \rho . \tag{29}$$

Thus substituting p(0) from (29) and $p(N,r) = 0$ in (23) we get the system-size distribution and the same substitutions in (25) yield the mean number of units. Thus, in this case, we have

$$G(w,r) = \sum_{m=1}^{\infty} p(m,r)\, w^m$$

$$= \frac{\lambda w (1-w)(1-\rho)\sigma_r}{A_r \left[w + \sum_{j=1}^{n} \frac{\sigma_j \mu_j}{A_j} \right]} \tag{30}$$

and

L = Mean number of units in the system

$$= \rho + \frac{\lambda^2 \sum_{r=1}^{n} \frac{\sigma_r}{\mu_r^2}}{1-\rho}$$

$$= \frac{\rho}{1-\rho} \left[1 - \rho + \lambda^2 \sum_{r=1}^{n} \frac{\sigma_r}{\mu_r^2} \right] \tag{31}$$

Pre-assigned Mean Service Rate:

If we are only given that the mean service rate over all the n service branches is μ, then a comparison between the existing models and the present model is possible. In order that the mean service rate be μ, we need only substitute $\mu_r = n\sigma_r \mu$ in the results already deduced above. We deduce the results below only when an infinite queue is allowed.

$$\mu_r = n\,\sigma_r\,\mu,$$

we find that

$$G(w) = \left\{ \frac{1}{1 + \rho \sum_{r=1}^{n} \frac{\sigma_r}{D_r}} - 1 \right\} w\,(1 - \rho) \qquad (32)$$

where

$$\rho = \lambda/\mu$$

and

$$D_r = \rho\,(w - 1) - n\,\sigma_r .$$

Therefore,

L = Mean number of units in the system

$$= \frac{\rho}{1-\rho}\left[1 - \rho + \frac{\rho \sum_{r=1}^{n} \frac{1}{\sigma_r}}{n^2}\right] \qquad (33)$$

$$> \frac{\rho}{1-\rho}$$

Also,

$L_q = L - \rho$ = Mean number of units in the queue

$$= \frac{\rho^2 \sum_{r=1}^{n} \frac{1}{\sigma_r}}{n^2(1-\rho)} > \frac{\rho^2}{1-\rho} .$$

We thus observe that in the case discussed, the fraction of time the service channel is empty is the same as in the queuing system M/M/1 while the mean number of units in the system and the queue increase.

Morse (1958) had discussed this particular case only when $n = 2$.

The Queuing System $M/IM-E_k/1$

As above, if we substitute

$$\sigma_r = \delta_{rk}$$

in (14), then we get the Laplace transform of the system size distribution in the transient case of the problem considered by Jain (1962).

Waiting Time Distribution of the Queuing System $M/H-IM-E_{nk}/1$

In the case of steady state, the Laplace-transform of the waiting time distribution can be evaluated. We evaluate it in this section.

Let $w(t)\,dt$ be the probability that a unit has to wait in the queue for a time between t and $t + dt$. Then, we have

$$\begin{aligned} w(t)\,dt = p(0)\,\delta(t)\,dt &+ \sum_{r=1}^{n} \sigma_r \mu_r e^{-\mu_r t} \sum_{s=1}^{k} \Big[p(1,s,r)(\mu_r t)^{s-1}/(s-1)! \\ &+ \sum_{s_1=0}^{k-1} p(2,s,r)(\mu_r t)^{s+s_1} d_{s_1+1}/(s+s_1)! \\ &+ \sum_{s_1=1}^{k-1} \sum_{s_2=1}^{k} p(3,s,r)\,(\mu_r t)^{s+s_1+s_2} d_{s_2} d_{s_1+1}/(s+s_1+s_2)! \\ &+ \cdots\cdots\cdots \Big] \end{aligned} \tag{34}$$

where $\delta(t)$ is the Dirac delta function.

Applying Laplace-transformation with parameter v to (34), we get

$$\bar{w}(v) = p(0) + \sum_{r=1}^{n} \frac{\sigma_r G(\theta_r, D(\theta_r), r)}{D(\theta_r)} \tag{35}$$

where

$$\theta_r = \frac{\mu_r}{\mu_r + v}.$$

Applying (18) to (10) we can calculate $G(w, z, r)$. Thus putting $w = \theta_r$ and $D(\theta_r)$ in that result, we get $G(\theta_r, D(\theta_r), r)$, which when substituted in (35) yields the required Laplace-transform of the waiting time distribution. Carrying through this procedure, we have

$$\overline{W}(v) = p(0) + \sum_{r=1}^{n} \left\{ \frac{\sigma_r}{\lambda(1 - \theta_r) D(\theta_r) + \mu_r \{D(\theta_r) - 1\}} \left[\lambda \theta_r^N (1 - \theta_r) \right. \right.$$

$$\sum_{s=1}^{k} p(N, s, r) \{D(\theta_r)\}^s - D\{D(\theta_r)\} \sigma_r \; p(0) \lambda (1 - \theta_r)$$

$$- \sum_{i=1}^{n} \left\{ \left(\delta_{ir} - \frac{\sigma_r \; D\{D(\theta_r)\}}{\theta_r} \right) \left((P_i - Q_i) + \frac{\sigma_i \; D(X_i) \sum_{j=1}^{n} (P_j - Q_j)}{\theta_r - \sum_{j=1}^{n} D(X_j) \sigma_j} \right. \right.$$

(36)

where

$$P_r = \lambda \theta_r^N (1 - \theta_r) \sum_{s=1}^{k} p(N, s, r) X_r^s$$

$$Q_r = D(X_r) \sigma_r p(0) \lambda (1 - \theta_r)$$

$$X_r = \frac{\mu_r}{\lambda(1 - \theta_r) + \mu_r}.$$

The mean waiting time can be calculated from (36) itself. In fact

$$\text{Mean waiting time} = - \frac{d}{dv} \overline{W}(v) \Big|_{v = 0}.$$

No Queue Case

In this model the arrival pattern, the queue discipline and the service time distribution are assumed to be the same as in the system $M/H-1M-E_{nk}/1$ described in the beginning of this section, but no queue in front of the service channel i

[illegible], i.e. $N = 1$. The continuity equations, assuming steady state conditions, [illegible]

$$\sum_{i=1}^{n} \mu_i \, p(1,1,i) - \lambda p(0) = 0 \tag{37}$$

$$\sigma_r \, d_s \sum_{i=1}^{n} \mu_i \, p(1,1,i) + \mu_r \, p(1,s+1,r) - \mu_r \, p(1,s,r) = 0 \tag{38}$$

The explicit solution of the above equations is given by

$$p(0) = \frac{1}{1+\rho} \tag{39}$$

$$p(1,1,r) = \frac{\lambda \sigma_r \sum_{s=1}^{k} (s \, d_s)}{\mu_r (1+\rho)} \tag{40}$$

where

$$\rho = \lambda \left\{ \sum_{s=1}^{k} (s \, d_s) \right\} \left\{ \sum_{r=1}^{n} \frac{\sigma_r}{\mu_r} \right\}.$$

Therefore,

$$P_{full} = \sum_{r=1}^{n} p(1,1,r) = \frac{\rho}{1+\rho} \tag{41}$$

$$P_{empty} = 1 - P_{full} = p(0) = \frac{1}{1+\rho} \tag{42}$$

Clearly for $\rho \leqslant 1$,

$$P_{full} \leqslant P_{empty} \tag{43}$$

[illegible] the system is busy is less than or equal to [illegible] [illegible] that the number, n, of [illegible] and the number, s, of phases does not affect the results.

As expected,

$$P_{full} \geqslant P_{empty} \tag{44}$$

if $\rho \geqslant 1$. Moreover, $P_{full} = P_{empty}$ when $\rho = 1$.

SECTION II: The Queuing System [illegible]-$E_{nk}/1$

The following problem is considered in this section:

Suppose that the arrival channel consists of n independent branches. Each branch consists of a fixed number, k, of phases in the reverse order, i.e. phase k is first and phase 1 is last. A unit arriving for service selects the branch and the number, s, $s = 1,2,\ldots,k$, of phases in that branch, which it chooses to enter the system. Let σ_r be the probability that the arriving unit enters from the rth branch, so that $\sum_{r=1}^{n} \sigma_r = 1$. Also let c_s be the probability that it requires s phases to pass before entering the system, so that $\sum_{s=1}^{k} c_s = 1$. Once a unit decides a particular branch to enter from and the number, s, of phases to pass in that branch, it has to pass through each of the phases one after the other. The time of staying in each phase of the rth branch is assumed to be distributed according to a Poisson distribution with parameter λ_r in all the phases of the rth branch. A unit enters the system (queue + service) from phase 1 only, again in accordance with the above distribution. We further assume that at any time only one unit can be there in the arrival channel, i.e. only one of the nk phases can be busy. We also assume the existence of a reservoir of infinite capacity attached to the arrival channel that emits a unit the moment the arrival channel is free. Thus the input distribution has the probability density a(t), where

$$a(t) = \sum_{s=1}^{k} \sum_{r=1}^{n} c_s \, \sigma_r \, \lambda_r \, e^{-\lambda_r t} \, \frac{(\lambda_r t)^{s-1}}{\underline{|s-1}}.$$

The queue discipline is assumed to be 'first come, first served'.

The units are finally served according to the exponential distribution with parameter μ, i.e. the service time distribution has the probability density function s(t), where

$$s(t) = \mu e^{-\mu t} .$$

The maximum number of units allowed in the system is N, so that when a unit comes from the arrival channel and finds N units in the system, it is not allowed to join the queue and thus it overflows, i.e. is lost to the system.

Continuity Equations and Their Solution

Let $p(m,s,r,t)$ denote the probability that at time t there are m units in the system, the unit in the arrival channel being in the sth phase of the rth branch. Assuming that there can be no more than N units in the system and observing that $p(m,k+1,r,t) = 0$ for all m and r, we have the following continuity equations:

$$\frac{d}{dt} p(m,s,r,t) = -(\lambda_r + \mu)p(m,s,r,t) + \lambda_r p(m,s+1,r,t) + \mu p(m+1,s,r,t) + \sigma_r c_s \sum_{i=1}^{n} \lambda_i p(m-1,1,i,t) \qquad (45)$$

$$(m = 1,2,\ldots,N-1)$$

$$\frac{d}{dt} p(N,s,r,t) = -(\lambda_r + \mu)p(N,s,r,t) + \lambda_r p(N,s+1,r,t) + \sigma_r c_s \sum_{i=1}^{n} \lambda_i \left[p(N-1,1,i,t) + p(N,1,i,t)\right] \qquad (46)$$

$$\frac{d}{dt} p(0,s,r,t) = -\lambda_r p(0,s,r,t) + \lambda_r p(0,s+1,r,t) + \mu p(1,s,r,t) \qquad (47)$$

where equations (45), (46) and (47) are valid for $s = 1,2,\ldots,k$ and $r = 1,2,\ldots,n$.

The system of differential difference equations (45), (46) and (47) represents $nk(N+1)$ independent equations in as many unknowns. To solve this system let us introduce the following generating functions:

$$F(m,z,r,t) = \sum_{s=1}^{k} p(m,s;r,t)\ z^s \tag{48}$$

$$G(w,z,r,t) = \sum_{m=0}^{N} F(m,z,r,t)\ w^m \tag{49}$$

$$H(w,1,r,t) = \sum_{m=0}^{N} p(m,1,r,t)\ w^m \tag{50}$$

and

$$C(z) = \sum_{s=1}^{k} c_s z^s \tag{51}$$

Multiplying equations (45), (46) and (47) by appropriate powers of z and w, adding, using (48), (49), (50), (51) and applying the Laplace transformation

$$\bar{f}(u) = \int_0^\infty e^{-ut} f(t)\ dt \tag{52}$$

we get

$$\left[u + \lambda_r(1 - 1/z) + \mu(1 - 1/w)\right] \bar{G}(w,z,r,u) - \sigma_r C(z)\ w^p - \sigma_r C(z)\ w \sum_{i=1}^{n}\left\{\lambda_i \bar{H}(w,1,i,u)\right\} + \lambda_r \bar{H}(w,1,r,u) - q(1 - w)\ C(z)\ w^N \sigma_r - \mu \bar{F}(0,z,r,u)(1-1/w) = 0 \tag{53}$$

where

$$q = \sum_{i=1}^{n}\left\{\lambda_i\ p(N,1,i,u)\right\}$$

and p (>0) is the initial number of units with which the system starts at time $t = 0$.

Let us substitute

$$z = \frac{\lambda_r w}{(u + \lambda_r + \mu)w - \mu} = X_r \quad \text{(say)}$$

in (53). Then, we get

$$\sum_{i=1}^{n}\left\{\left[\delta_{ir} - w\sigma_r C(X_r)\right]\lambda_i \overline{H}(w,1,i,u)\right\} = P_r + Q_r \tag{54}$$

where

$$P_r = \mu\,(1 - 1/w)\sum_{s=1}^{k}\left\{\overline{p}(0,s,r,u)\; X_r^{\,s}\right\} + \sigma_r\, w^p\, C(X_r),$$

$$Q_r = q(1 - w)\;\sigma_r\; C(X_r)\; w^N.$$

Equation (54) can be written as the matrix equation

$$\underset{\sim}{B}\underset{\sim}{H} = \underset{\sim}{C} \tag{55}$$

where $\underset{\sim}{B}$ is the matrix given by

$$\underset{\sim}{B} = \left\|b_{ij}\right\|$$

such that

$$b_{ij} = -w\, C(X_i)\;\sigma_i\;\;\lambda_j \qquad (i \neq j)$$

$$b_{ii} = \lambda_i - w\, C(X_i)\,\sigma_i\,\lambda_i$$

and $\underset{\sim}{H}$ and $\underset{\sim}{C}$ are the column matrices

$$\left[\overline{H}(w,1,1,u),\ \overline{H}(w,1,2,u),\ldots\ldots,\overline{H}(w,1,n,u)\right]$$

and

$$\left[P_1 + Q_1,\ P_2 + Q_2,\ldots\ldots\ldots\ldots,\ P_n + Q_n\right]$$

respectively.

We observe that $\underset{\sim}{B}^{-1}$ is given by

$$\underset{\sim}{B}^{-1} = \left\|b'_{ij}\right\|$$

such that

$$b'_{ij} = \frac{C(X_i)\ w\ \sigma_j}{\lambda_i\ S} \qquad (i \neq j)$$

and $$b'_{ii} = \frac{S + w\ C(X_i)\ \sigma_i}{\lambda_i\ S}$$

where

$$S = 1 - w \sum_{j=1}^{n} C(X_j)\ \sigma_j .$$

When this value of $\underset{\sim}{B}^{-1}$ is substituted in (55), we get

$$\overline{H}(w,1,r,u) = \frac{P_r + Q_r}{\lambda_r} + \frac{w\ C(X_r)\ \sigma_r \sum_{j=1}^{n} (P_j + Q_j)}{\lambda_r\ S} \tag{56}$$

The right hand side of (56) contains nk + 1 unknown probabilities, viz. q and $\bar{p}(0,s,r,u)$ for $s = 1,2,\ldots,k$ and $r = 1,2,\ldots,n$. Also the denominator on the right hand side is a polynomial of order nk + 1. Since on the left hand side we have a polynomial, we get nk + 1 equations as conditions of analyticity of the right hand side. These nk + 1 equations determine the nk + 1 unknowns and thus $\overline{H}(w,1,r,u)$ is determined for all r.

Substituting back these values of $\overline{H}(w,1,r,u)$ and the unknowns determined in (54), we can determine $\overline{G}(w,z,r,u)$.

Let us now introduce another generating function

$$\overline{G}(w,1,u) = \sum_{r=1}^{n} \overline{G}(w,1,r,u) \tag{57}$$

Substituting the values of $\overline{G}(w,1,r,u)$ determined above in (57), we get

$$\overline{G}(w,1,u) = \frac{1}{u + \lambda(1 - 1/w)} \Bigg[w^{P} + \sum_{r=1}^{n} \sum_{i=1}^{n} \Bigg\{ \Bigg(\frac{w\ \sigma_r \{1 - C(X_r)\}}{1 - w \sum_{j=1}^{n} C(X_j)\ \sigma_j} - \delta_{ir} \Bigg) (P_i + \ldots$$

$$+ q(1 - w)\ w^{N} + \lambda (1 - 1/w) \sum_{r=1}^{n} \sum_{s=1}^{k} \bar{p}(0,s,r,u) \Bigg] \tag{58}$$

This is the required function representing the Laplace transform of system-size distribution from which the queue parameters can be calculated. For example,

Laplace transform of the mean number of units in the system

$$= \frac{d}{dw} \bar{G}(w,1,u) \Big|_{w=1}$$

$$= \frac{1}{u}\Bigg[-\mu \bar{G}(1,1,u) + \mu \sum_{r=1}^{n}\sum_{s=1}^{k} \bar{p}(0,s,r,u) - q + p + \frac{\sum_{j=1}^{n} \sigma_j C(Y_j)}{1 - \sum_{j=1}^{n} C(Y_j)\sigma_j} \left\{ \sum_{r=1}^{n} \sigma_r (1 - C(Y_r)) \left(1 + \frac{\sum_{j=1}^{n}\left\{\sigma_j C(Y_j) - L_j\right\}}{1 - \sum_{j=1}^{n}\sigma_j C(Y_j)} \right) + \sum_{r=1}^{n} \sigma_r L_r \right\} +$$

$$\sum_{i=1}^{n} \sum_{r=1}^{n} \left\{ \left(\frac{\sigma_r \left\{ 1 - C(Y_r) \right\}}{1 - \sum_{j=1}^{n} \sigma_j C(Y_j)} - \delta_{ir} \right) \left(\mu \sum_{s=1}^{k} \bar{p}(0,s,i,u)\, Y_i^{s} + \sigma_i C(Y_i)\, p - \sigma_i L_i - \sigma_i C(Y_i)\, q \right) \right\} \Bigg] \tag{59}$$

where

$$Y_r = \frac{\lambda_r}{u + \lambda_r}$$

$$L_r = \frac{\mu \sum_{s=1}^{k} (s\, c_s\, Y_r^{s})}{u + \lambda_r}.$$

Steady State Solution

In this section we deduce the steady state solution of the problem as a limiting case of the results arrived at in the previous section. Applying Abel's corollary to (58), we get the solution in this case as given below:

$$G(w,1) = \frac{1}{\mu(1 - 1/w)}\left[\sum_{r=1}^{n}\sum_{i=1}^{n}\left\{\left(\frac{w\,\sigma_r\{1 - C(X_r)\}}{1 - w\sum_{r=1}^{n} C(X_r)\,\sigma_r} - \delta_{ir}\right)(P_i + Q_i)\right\} + q(1 - w)w^N + \mu(1 - 1/w)\sum_{r=1}^{n}\sum_{s=1}^{k} p(0,s,r)\right] \qquad (60)$$

where now

$$X_r = \frac{\lambda_r w}{(\lambda_r + \mu)w - \mu}$$

$$P_r = \mu(1 - 1/w)\sum_{s=1}^{k}\left\{p(0,s,r)\,X_r^{\,s}\right\}$$

$$Q_r = q(1 - w)\,\sigma_r\,C(X_r)\,w^N$$

and

$$q = \sum_{i=1}^{n}\left\{\lambda_i\,p(N,1,i)\right\}.$$

Infinite Queue Case (H-$1M$-$E_{nk}/M/1$)

In this case $\overline{p}(N,s,r,u) \longrightarrow 0$ as $N \longrightarrow \infty$, i.e. $q \longrightarrow 0$ as $N \longrightarrow \infty$. Thus there remain nk unknowns on the right hand side of (56). Also there is an infinite series on the left hand side which must converge within the unit circle at least. We can prove by using Rouche's Theorem that nk roots of the denominator lie inside the unit circle and one root lies outside. Thus we get nk equations due to analyticity which are sufficient to determine the unknowns.

The same substitutions in (58) and (59) yield the corresponding results in this case.

Queuing System $M\text{-}H_r/M/1$

In this section, we deduce the results obtained by Gupta and Goyal ( ).

Substituting

$$c_s = \delta_{s1}$$

in (58), we get

$$\ldots(w) = \frac{1}{u + \mu(1 - 1/w)} \left[w^r + \sum_{r=1}^{n} \sum_{i=1}^{n} \left\{ \left(\frac{w\,\sigma_r(1 - X_r)}{1 - w\sum_{j=1}^{n}\sigma_j X_j} - \delta_{ir} \right) (P_i + Q_i) \right\} + q(1 - w)\,w^N + \mu(1 - 1/w) \sum_{r=1}^{n} \bar{p}(0,r,u) \right] \quad (61)$$

where

$$P_r = \mu(1 - 1/w)\,\bar{p}(0,r,u)\,X_r - \sigma_r w^p X_r$$

$$Q_r = q(1 - w)\,\sigma_r X_r w^N .$$

By applying Abel's corollary to (53) and substituting $c_s = \delta_{s1}$, the steady state solution of the queuing system $H\text{-}M_r/M/1$ is obtained and is given by

$$G_r(w) = \frac{\mu(1 - w)\,p(0,r)}{w\,B_r} - \frac{\sigma_r}{B_r\left[1 + \mu\sum_{j=1}^{n}(\sigma_j/B_j)\right]} \left[\mu\sum_{j=1}^{n}\frac{\lambda_j\,p(0,j)}{B_j} + k\,w^N\right] \quad (62)$$

where $p(m,r)$ denotes the equilibrium probability that there are m units in the system, the unit in the arrival channel being in the rth branch,

$$G_r(w) = \sum_{m=0}^{N} p(m,r)\,w^m,$$

$$B_r = \frac{\mu(1 - w)}{w} - \lambda_r .$$

$$k = \sum_{j=1}^{n} \lambda_j \, p(N,j).$$

The equations that we obtain in view of the analyticity of (62) are

$$\sum_{r=1}^{n}\left[\sigma_r \lambda_r \left(\prod_{\substack{j=1\\ j\neq r}}^{n} B_j\right)\right] - \mu \sum_{r=1}^{n}\left[\lambda_r \, p(0,r) \left(\prod_{\substack{j=1\\ j\neq r}}^{n} B_j\right)\right] = 0 \qquad (63)$$

$$B_r = \frac{\mu(1 - w_i)}{w_i} - \lambda_r$$

w_i being the n roots of

$$\prod_{r=1}^{n} B_r + w \sum_{r=1}^{n}\left(\sigma_r \lambda_r \prod_{\substack{j=1\\ j\neq r}}^{n} B_j\right) = 0.$$

$$G(w) = \sum_{r=1}^{n} G_r(w) = \frac{\mu(1-w)}{w} \sum_{r=1}^{n} \frac{p(0,r)}{B_r} - \frac{\sum_{r=1}^{n}(\sigma_r/B_r)}{1 + \mu\sum_{r=1}^{n}(\sigma_r/B_r)}\left[\mu \sum_{r=1}^{n} \lambda_r \, p(0,r)/B_r - k\, w^N\right] \qquad (64)$$

The normalizing equation

$$F(1) = 1$$

gives

$$p(0) - k/\mu = 1 - \rho \qquad (65)$$

where

$$p(0) = \sum_{r=1}^{n} p(0,r)$$

$$\rho = \left[\mu \sum_{r=1}^{n} (\sigma_r/\lambda_r)\right]^{-1}.$$

The n equations represented by (63) together with (65) are sufficient to determine the n + 1 unknowns p(0,r) and k.

Hence $G(w)$, given in (64), is completely known. This is the required function giving the system size distribution. We may now calculate the mean number of units in the system, M, from (64) by using the formula

$$M = \frac{d}{dw} G(w) \Big|_{w=1}$$

and is given by

$$M = \mu \sum_{r=1}^{n} \frac{p(0,r)}{\lambda_r} - \frac{\rho^2 \mu \{\mu p(0) + k\}\left\{\sum_{r=1}^{n} (\sigma_r / \lambda_r^2)\right\}}{(\rho - 1)^2} + \frac{\rho \sum_{r=1}^{n} (\sigma_r / \lambda_r) \left[\mu^2 \sum_{r=1}^{n} p(0,r)/\lambda_r - Mk\right]}{\rho - 1} \tag{66}$$

In case an infinite queue is allowed, then we have simply to put $k = 0$ in (64) and (66) to obtain the function representing the system size distribution and the mean number of units in the system. Also, we may observe that the normalizing equation (65) in this case becomes

$$p(0) = 1 - \rho \tag{67}$$

which shows that the probability, $p(0)$, in the case of infinite queue permissible is insensitive to the change in the arrival time distribution.

Pre-Assigned Mean Arrival Rate

If, however, we insist that the overall mean arrival rate is preassigned as λ, then we can deduce the results simply by substituting $\lambda_r = n\sigma_r\lambda$ in the results deduced in the above sub-section. Substituting this value in (64), we have in the infinite queue permissible case, i.e. when $k = 0$

$$G(w) = w(1 - 1/w) \sum_{r=1}^{n} \frac{p(0,r)}{C_r} - \frac{n\rho \sum_{r=1}^{n} (\sigma_r/C_r) \left[\sum_{r=1}^{n} (\sigma_r p(0,r)/C_r)\right]}{1 + \sum_{r=1}^{n} (\sigma_r/C_r)} \tag{68}$$

where

$$C_r = \frac{1-w}{w} - n\sigma_r \rho ,$$

$$\rho = \lambda/\mu .$$

Also, these substitutions into (66) yield

$$M = \frac{\sum_{r=1}^{n} \left(p(0,r)/\lambda_r\right)}{n(\rho - 1)} - \frac{p(0) \sum_{r=1}^{n} (1/\sigma_r)}{\left[n(\rho - 1)\right]^2} \tag{69}$$

Morse (1958) had deduced results corresponding to (68) and (69) when $n = 2$.

No Queue Case ($H-1M-E_{nk}/1$)

If no queue is allowed in front of the service channel, then the queue equations are given by (46) and (47) when $N = 1$. In steady state case, the explicit solution of the equation thus obtained is given by

$$F(0,1,r) = \sigma_r \left[1/\lambda_r - \frac{1 - C(Z_r)}{\mu \sum_{s=1}^{k} (s\, c_s)} \right] \left[\sum_{j=1}^{n} (\sigma_j/\lambda_j) \right]^{-1} \tag{70}$$

and

$$F(1,1,r) = \frac{\sigma_r \left[1 - C(Z_r)\right]}{\mu \sum_{j=1}^{n} (\sigma_j/\lambda_j) \left[\sum_{s=1}^{k} (s\, c_s)\right]} \tag{71}$$

where

$$Z_r = \frac{\lambda_r}{\mu + \lambda_r} .$$

In (70) and (71), if we substitute

$$c_s = \delta_{s1},$$

we get

$$p(0,r) = \frac{\mu\, \sigma_r}{\lambda_r(\mu + \lambda_r)\sum_{j=1}^{n}(\sigma_j/\lambda_j)} \tag{72}$$

and

$$p(1,r) = \frac{\sigma_r}{(\mu + \lambda_r)\sum_{r=1}^{n}(\sigma_r/\lambda_r)} \tag{73}$$

Now,

$$P_{lost} = A \sum_{r=1}^{n} \sigma_r\, p(1,r)$$

and

$$P_{served} = A \sum_{r=1}^{n} \sigma_r\, p(0,r),$$

where A is some constant to be determined.

Using the fact that

$$P_{served} + P_{lost} = 1,$$

we can determine A, and hence we get

$$P_{lost} = \left[\sum_{r=1}^{n} \frac{\sigma_r^2}{\mu + \lambda_r}\right]\left[\sum_{r=1}^{n} \frac{\sigma_r^2}{\lambda_r}\right]^{-1} \tag{74}$$

and

$$P_{served} = \left[\sum_{r=1}^{n} \frac{\mu\, \sigma_r^2}{(\mu + \lambda_r)\lambda_r}\right]\left[\sum_{r=1}^{n} \frac{\sigma_r^2}{\lambda_r}\right]^{-1}. \tag{75}$$

If $\lambda_r = n\, \sigma_r \lambda$, then some comparison with the known results is possible. For instance,

$$P_{empty} = \sum_{r=1}^{n} p(0,r) = \frac{1}{n}\sum_{r=1}^{n} \frac{1}{1 + n\, \sigma_r\, \rho}$$

$$> \frac{1}{1+\rho}, \tag{76}$$

and

$$P_{full} = 1 - P_{empty} < \frac{\rho}{1+\rho} . \tag{77}$$

Thus the fraction of time the service channel is empty is greater and the fraction of time the service channel is busy is lesser in this case than the corresponding characteristics in the case of M/M/1 queuing system.

Also clearly,

$$P_{served} \leq \frac{1}{1+\rho}, \tag{78}$$

and

$$P_{lost} \geq \frac{\rho}{1+\rho} . \tag{79}$$

Thus the fraction of units lost is greater and consequently the fraction of units served is lesser in this case than in the case of M/M/1 queuing system.

We can further show that P_{lost} is maximum when

$$\sigma_r = \delta_{rn},$$

i.e. when only one channel is operative.

We suppose that

$$\sigma_n = 1, \quad \sigma_i = 0 \text{ for } i = 1,2,\ldots,n-1.$$

Let

$$\sum_{r=1}^{n} \frac{1}{1 + n\sigma_r \rho} = S,$$

so that

$$S = \left[n + n\rho(n-1) + n^2\rho^2(n-2)\sum\sigma_1\sigma_2 + n^3\rho^3(n-3)\sum\sigma_1\sigma_2\sigma_3 + \cdots\cdots + (n\rho)^{n-1}\sum\sigma_1\sigma_2\sigma_3\cdots\sigma_{n-1}\right] \cdot \left[1 + n\rho + n^2\rho^2\sum\sigma_1\sigma_2 + n^3\rho^3\sum\sigma_1\sigma_2\sigma_3 + \cdots\cdots + (n\rho)^{n-1}\sum\sigma_1\sigma_2\cdots\sigma_{n-1} + (n\rho)^n\sigma_1\sigma_2\sigma_3\cdots\sigma_n\right]^{-1}.$$

When $\sigma_i = 0 (i = 1,2,\ldots,n-1)$ and $\sigma_n = 1$, this expression becomes

$$S' = \frac{n + (n-1)n\rho}{1 + n\rho}$$

To show that $S' > S$, we use the fact that

$$x/y > (x + a)/(y + b),$$

all quantities being positive if

$$bx - ay > 0,$$

so that

$$S' > S$$

if

$$\{n + (n-1)n\rho\}\{n^2\rho^2\sum\sigma_1\sigma_2 + n^2\rho^3\sum\sigma_1\sigma_2\sigma_3 + \cdots\cdots\cdots + (n\rho)^{n-1}\sum\sigma_1\sigma_2\cdots\cdots\cdots\sigma_{n-1} + (n\rho)^n\sigma_1\sigma_2\sigma_3\cdots\cdots\sigma_n\} -$$

$$(1 + n\rho)\{n^2\rho^2(n-2)\sum\sigma_1\sigma_2 + n^3\rho^3(n-3)\sum\sigma_1\sigma_2\sigma_3 + \cdots\cdots\cdots + (n\rho)^{n-1}\sum\sigma_1\sigma_2\sigma_3\cdots\cdots\sigma_{n-1}\} > 0,$$

or

$$n^2\rho^2\sum\sigma_1\sigma_2\{n + (n-1)n\rho - (1 + n\rho)(n-2)\} +$$
$$n^3\rho^3\sum\sigma_1\sigma_2\sigma_3\{n + (n-1)n\rho - (1 + n\rho)(n-3)\} + \cdots\cdots\cdots$$
$$(n\rho)^{n-1}\sum\sigma_1\sigma_2\sigma_3\cdots\cdots\sigma_{n-1}\{n + (n-1)n\rho - (1 + n\rho)\} +$$
$$(n\rho)^n\sigma_1\sigma_2\sigma_3\cdots\cdots\cdots\sigma_n > 0,$$

or

$$n^2\rho^2\sum\sigma_1\sigma_2(2 + n\rho) + n^3\rho^3\sum\sigma_1\sigma_2\sigma_3(3 + 2n\rho) + \cdots\cdots\cdots +$$
$$(n\rho)^r\sum\sigma_1\sigma_2\sigma_3\cdots\cdots\sigma_{r-1}\{r + (r-1)n\rho\} + \cdots\cdots\cdots +$$

$$(n\rho)^{n-1}\sum \sigma_1\sigma_2\sigma_3\cdots\cdots\sigma_{n-1}\left\{(n-1)+(n-2)n\rho\right\} +$$

$$(n\rho)^n\sigma_1\sigma_2\sigma_3\cdots\cdots\sigma_n > 0,$$

which is true, since all the quantities involved are positive. Since $S' > S$ the value of P_{lost} is maximum when $\sigma_r = \delta_{rn}$. Also,

$$\max P_{lost} = \frac{n\rho}{1+n\rho}. \qquad (60)$$

The Queuing System $1M\text{-}E_k/M/1$

As in the section I, if we substitute

$$\sigma_r = \delta_{rk}$$

in (58), we get the results due to Jaiswal (1961).

SECTION III: The Queuing System $M/H\text{-}G_n/1$

In this section an attempt is made to introduce general distributions in each of the n branches of the service channel. The technique of introducing supplementary variables as illustrated by Cox (1955), is used to render the process Markovian. The analysis of the solution is more or less similar to the one as given by Keilson and Kooharian (1960), who considered the queuing system M/G/1. One deviation here from Keilson and Kooharian (1960) is that our phase space consists of the elements of the Cartesian product $J \times N \times R$ instead of $J \times R$ where R is the set of non-negative real numbers, J is the set of natural numbers including zero and N is a finite subset of J, viz.

$\{1, 2, \ldots, n\}$.

The following problem has been considered in this section:

Suppose that the units arrive at a service facility according to a stationary Poisson stream with parameter λ. The service facility provides n types of services at the n branches of a service channel, all the n branches being attended to by the same operator. The service time x counted from admission to completion in the rth branch of the service facility is specified by an arbitrary probability density function $D_r(x)$, say. An arriving unit joins the service facility immediately if no one is being served there, otherwise the arriving units go on forming a queue in front of the service channel in order of their arrival. The moment the service of a unit being served is completed, the unit at the head of the queue joins the rth branch of the service channel with probability σ_r, so that $\sum_{r=1}^{n}\sigma_r = 1$.

Since our phase space is $J \times N \times R$, as pointed out above, we introduce the notation $p(m,r,x,t)$ to denote the probability that at time t there are m units in the queue (excluding servee), the unit in the service facility being in the rth branch with elapsed service time x. In addition to these probabilities, let $E(t)$ denote the probability that at time t the system (queue + service) is in complete vacant state.

Our problem is to calculate the probabilities introduced above.

<u>Continuity Equations and Their Solution</u>

To facilitate the writing of continuity equations, let us introduce the probabilities $\eta_r(x)\Delta$ as the first order probability that a service completion occurs in the rth branch of the service channel in the interval $(x, x+\Delta)$, given that the unit was not served upto time x. $\eta_r(x)$ is related to $D_r(x)$ by the relation

$$D_r(x) = \eta_r(x) \exp\left[-\int_0^x \eta_r(u)\, du\right]. \tag{81}$$

As in Keilson and Kooharian (1960), we have the following continuity equations valid for $r = 1,2,\ldots,n$:

$$D\left[p(m,r,x,t)\right] = \lambda p(m-1,r,x,t) \tag{82}$$

$$(m = 1,2,\ldots\ldots)$$

$$D\left[p(0,r,x,t)\right] = 0 \tag{83}$$

$$\frac{d}{dt} E(t) + \lambda E(t) = \sum_{s=1}^{n}\left[\int_0^\infty \eta_s(x)\; p(0,s,x,t)\; dx\right] \tag{84}$$

where the operator D stands for

$$\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda + \eta_r(x).$$

These continuity equations are to be solved under the boundary conditions

$$p(m,r,0,t) = \sigma_r \sum_{s=1}^{n}\left[\int_0^\infty p(m+1,s,x,t)\,\eta_s(x)\; dx\right] \tag{85}$$

$$p(0,r,0,t) = \sigma_r \sum_{s=1}^{n}\left[\int_0^\infty p(1,s,x,t)\;\eta_s(x)\; dx\right] + \lambda\,\sigma_r\, E(t) \tag{86}$$

and the initial condition

$$G(y,r,x,0) = 0,\; E(0) = 1 \tag{87}$$

where

$$G(y,r,x,t) = \sum_{m=0}^{\infty} p(m,r,x,t)\; y^m.$$

It may be remarked here that we have chosen a special case of the initial conditions in (87) to facilitate the analysis, otherwise in the general case, the initial conditions are specified by

$$p(m,r,x,0) = \delta_{mN}\ \delta_{rk}\ \delta(x - x_o) \tag{88}$$

$$(m = 0,1,\ldots;\ r = 1,2,\ldots n)$$

where δ_{mN} and δ_{rk} are Kronecker deltas and $\delta(x - x_o)$ is the Dirac delta function.

The solution of equations (82) and (83) as given by Keilson and Koocharian (1960) is

$$H(y,r,x,t) = H_o(y,r,t-x)\ \exp\left[-\lambda(1-y)\ x\right] \tag{89}$$

where

$$H(y,r,x,t) = G(y,r,x,t)\ \exp\left[\int_0^x \eta_r(u)\ du\right] \tag{90}$$

and $H_o(y,r,t-x)$ are n functions to be determined later from the boundary conditions.

Adding equations (84), and (85) and using (81), (89) and (90) and taking the Laplace transformation with parameter p, we get the equation

$$y\ \overline{H}_o(y,r,p) - \sigma_r \sum_{s=1}^{n} \overline{D}_s(q)\ \overline{H}_o(y,s,p) = \left[1 - q\ \overline{E}(p)\right]\sigma_r \tag{91}$$

which may be written in the matrix form as

$$\underset{\sim}{AH} = \left[1 - q\ \overline{E}(p)\right] \underset{\sim}{B} \tag{92}$$

where $\underset{\sim}{A}$ is the matrix

$$\underset{\sim}{A} = \| a_{ij} \|$$

such that

$$a_{ii} = y - \sigma_i\ \overline{D}_i(q)$$

$$a_{ij} = -\sigma_i\ \overline{D}_j(q) \qquad (i \neq j)$$

where

$$q = p + \lambda(1 - y)$$

and $\underset{\sim}{H}$ and $\underset{\sim}{B}$ are respectively the column matrices

$$\left[\overline{H}_0(y,1,p),\ \overline{H}_0(y,2,p),\ldots\ldots\ldots,\overline{H}_0(y,n,p)\right]$$

and

$$\left[\sigma_1,\ \sigma_2,\ldots\ldots\ldots\ldots\ \sigma_n\right]$$

and the bars denote Laplace transformations of the corresponding functions with parameters enclosed within parenthesis, e.g.

$$\overline{f}(p) = \int_0^{\infty} e^{-px}\, f(x)\, dx.$$

Observing that $\underset{\sim}{A}^{-1}$ is given by

$$\underset{\sim}{A}^{-1} = \left\| a'_{ij} \right\|$$

such that

$$a'_{ii} = \frac{R + \sigma_i\, \overline{D}_i(q)}{Ry}$$

$$a'_{ij} = \frac{\sigma_i\, \overline{D}_j(q)}{Ry} \qquad (i \neq j)$$

where

$$R = y - \sum_{j=1}^{n} \sigma_j\, \overline{D}_j(q),$$

we have from (92)

$$\overline{H}_0(y,r,p) = \frac{\sigma_r\left[1 - q\, \overline{B}(p)\right]}{R} \qquad (93)$$

$$(r = 1,2,\ldots,n)$$

Since now $\overline{H}_0(y,r,p)$ must be analytic, we have to so choose $\overline{E}(p)$ that the numerator of (93) cancels the zeros of the denominator. Arguing as in Keilson and Kooharian (1960), we find that if

$$\overline{E}(p) = \left[p + \lambda(1 - y_p)\right]^{-1} \tag{94}$$

then $\overline{H}_0^{*}(y,r,p)$ becomes analytic, where now

$$y_p = \sum_{j=1}^{n} \sigma_j \overline{D}_j \left[p + \lambda(1 - y_p)\right] \tag{95}$$

It must be admitted here that the inversion of (93) even in the simplest case, i.e. when $D_j(x) = \mu_j e^{-\mu_j x}$, is tedious enough and is not attempted here. However, the steady state case can be easily deduced from above by using standard arguments. This case is considered next.

Steady State Solution

In this section, we deduce the steady state solution of the problem under consideration. The mean number of units in the system is also calculated.

$$\lim_{t \to \infty} E(t) = \lim_{p \to 0+} p\,\overline{E}(p) = \lim_{p \to 0+} \frac{p}{p + \lambda(1 - y_p)}$$

$$= \frac{1}{1 - \lambda \left.\frac{dy_p}{dp}\right|_{p=0}} = 1 - \rho \tag{96}$$

where

$$\rho = \lambda \sum_{j=1}^{n} (\sigma_j / \eta_j),$$

$$\eta_j = \left[\int_0^{\infty} x\, D_j(x)\, dx\right]^{-1}.$$

Similarly,

$$\lim_{t\to\infty} H_o(y,r,t) = \lim_{p\to 0+} p\,\overline{H}_o(y,r,p) = \frac{\sigma_r\,\lambda(y-1)(1-\rho)}{y-\sum_{j=1}^{n}\sigma_j\,\overline{D}_j\left[\lambda(1-y)\right]}$$

Thus in this case, we have

$$G(y,r,x) = \frac{1}{y-\sum_{j=1}^{n}\sigma_j\,\overline{D}_j\left[\lambda(1-y)\right]}\left[\lambda\sigma_r(y-1)(1-\rho)\exp\left\{-\lambda(1-y)x-\int_0^x \eta_r(u)\,du\right\}\right] \tag{98}$$

$$\text{Mean number of units in the system} = \int_0^\infty\left[\sum_{r=1}^{n}\left\{\frac{d}{dy}G(y,r,x)\right\}\Big|_{y=1}\right]dx$$

$$= \lambda^2\int_0^\infty\left[\left\{x-\frac{\lambda\sum_{j=1}^{n}\sigma_j\int_0^\infty x^2 D_j(x)\,dx}{2(1-\rho)}\right\}\sum_{r=1}^{n}\left\{\sigma_r\exp\left(-\int_0^x\eta_r(u)\,du\right)\right\}\right]dx \tag{99}$$

<u>The Queuing System $M/H\text{-}D_r/1$</u>

Substituting

$$D_j(x) = 1/k_j \qquad 0\leq x\leq k_j$$
$$= 0 \qquad x > k_j$$
$$(j = 1,2,\ldots,n)$$

in (98), we get

$$G(y,r) = \int_0^\infty G(y,r,x)\,dx$$

$$= \frac{\lambda (y-1)(1-\rho)\sigma_r \left[1 + \left(\exp\{-\lambda(1-y)x_r\} - 1 \right) \Big/ k_r \right]}{\lambda \rho (y-1) + \sum_{j=1}^{n} \frac{\sigma_j}{k_j} \left[\exp\{-\lambda(1-y)x_r\} - 1 \right]} \tag{100}$$

The Queuing System $M/M-M_n/1$ (infinite waiting space)

Substituting

$$D_j(x) = \mu_j e^{-\mu_j x}$$

in (98) and integrating over $0 \leq x \leq \infty$ we easily get the result obtained by Gupta and Goyal (1964a), reproduced already in section I of this chapter, viz. equation (30).

The Queuing System M/G/1.

Substituting

$$\sigma_r = \delta_{rk} \text{ and } D_k(x) = D(x)$$

in (93), we get

$$\bar{H}_o(y,p) = \frac{[p + \lambda(1-y)]\bar{E}(p) - 1}{\bar{D}[p + \lambda(1-y)] - y}$$

which verifies a result obtained earlier by Keilson and Kooharian (1960).

CHAPTER IV

QUEUES WITH STATE DEPENDANT PARAMETERS

In most of the queuing models investigated so far, it is assumed that the sequences of random variables representing the inter-arrival times, $\{t_n\}$, and the service times, $\{s_n\}$, are independently and identically distributed and that these two sequences are also distributed independently. Another assumption, usually made about these sequences is that the mean arrival and service rates are constant.

Loynes (1962) has, however, considered queues in which the service time of the nth customer may depend upon the inter-arrival time $t_{n+1} - t_n$.

The other restriction, mean arrival and service rates being constant, can be relaxed in two apparently different ways. One way is to assume that these queue parameters depend upon 'time' and the other is to assume that these parameters depend upon the state of the system. However, very little work seems to have been done in either of these directions.

As far as queues with 'time dependant' parameters are concerned, the papers of Clark (1956) and Luchak (1956, 1957) are especially notable. Clark (1956) considered the queuing system M/M/1 with time-dependant parameters and later Luchak (1957) introduced time dependant arrival rate in the queuing system with Poisson input and wide variety of service time distributions.

As far as queues with state dependant arrival and/or service rates are concerned, one finds the papers of Haight (1957, 1959, 1960); Ancker and Gafarian (1962, 1963, 1963a, 1963b); Hillier, Conway and Maxwell (1964); Conway and Maxwell (1961); Yotam (1956) etc., who have all studied queues under equilibrium conditions. Haight (1957) found out the stationary state

distribution for the queuing system M/M/1 with balking by assuming various balking probability distributions, i.e. he considered the state dependant arrival rate. Ancker and Gafarian (1963, 1963a) studied the queuing system M/M/1 with balking and reneging, i.e., both mean arrival and service rates depending upon the state of the system, and again (1963b), they studied the queuing system M/M/s with reneging, servers being heterogeneous, i.e. only the mean service rates depending upon the state of the system. Hiller, Conway and Maxwell (1964) also studied the queuing system M/M/s with state dependant service rates. Conway and Maxwell (1961) studied the queuing system M/M/1 with state dependant service rate and also proposed a more general model in which both the arrival and service rates depend upon the state of the system. Votaw (1956) considered a queuing system which has one or more service stations, one or more sources which generate units to be served at the service stations, and a certain number of waiting places, i.e. only the arrival rate depending upon the state of the system. He assumed the input distribution to be Poisson and (i) service time distribution at a number of service stations to be exponential, (ii) constant service time distribution at one station. Saaty (1961) has also posed many such problems (at the end of chapter 4) for the queuing system with Poisson input and exponential service time distribution.

The purpose of the present chapter is to introduce state dependent arrival and service rates in the queuing systems (i) $M/H\text{-}M_K/1$ and (ii) $H\text{-}M_K/M/1$, the functions defining the dependance of the mean arrival and service rates upon the state of the system being assumed to be arbitrary. These two systems are studied in sections I and II respectively. Obviously, the results of all the papers quoted above for the queuing system M/M/1 can be deduced as particular cases from the present study as we are introducing arbitrary state

dependant mean arrival and service rates and also changing the arrival time distribution or the service time distribution which simulates the Poisson distribution as a particular case. One obvious advantage of introducing arbitrary functions defining the dependance of the mean arrival and service rates upon the state of the system, particularly in the problems considered in this chapter, is that we may introduce different types of functions, if need be there, in the different branches of the arrival or service channel.

To solve the problems formulated, we no longer can use the powerful technique of generating functions and thus resort to a rather heuristic type of technique which enables us to obtain a recurrence relation connecting the various probabilities from which queue characteristics may be evaluated numerically only when the maximum queue allowed is finite. To show the workability of the procedure outlined, we assume some particular forms of the state functions and draw the graphs for some of the queue characteristics. It must, of course, be admitted that the solution sought is computer oriented.

SECTION I: The Queuing System $M/H-E_k/1$ with state dependant parameters

The following problem is considered in this section:

Suppose that units arrive at random at a service facility consisting of k independent branches, i.e., which offers k types of services, according to a Poisson distribution with parameter $\lambda(n)$ when there are n units in the system (queue + service), i.e. the first order probability that in time dt a unit arrives is $\lambda(n)$ dt when there are already n units in the system. The service channel is busy if a unit is present in any one of the k branches of the service channel and in case the service channel is busy, the arriving units go on forming a queue in order of their arrival, the maximum number of units

which the system can accomodate being N. The moment the service channel disposes of the unit being served, the unit at the head of the queue joins any one of the k branches of the service channel. The fraction of time (on the average) it goes to the rth branch is σ_r, so that $\sum_{r=1}^{k} \sigma_r = 1$. The service time distribution in the rth branch of the service channel is exponential with parameter $\mu_r(n)$ when there are n units in the system, i.e., the first order probability that a unit is served in the rth branch of the service channel in time dt is $\mu_r(n)$ dt when there are n units in the system.

Obviously, if $\lambda(n) = \lambda$; $\mu_r(n) = \mu_r$ for all n, then the problem as formulated above reduces to the problem considered earlier by Gupta and Goyal (1964a), reported already as a particular case in section I of chapter III of this thesis.

Continuity Equations and Their Solution

Let p(n,r) denote the probability that there are n units in the system, the unit being served being in the rth branch of the service channel. Also let p(0) be the probability that the system is empty. Thus our phase space consists of the elements of the Cartesian product $I \times J$ and the point corresponding to p(0) where I and J are the finite sets $I = \{1, 2, \ldots, N\}$, $J = \{1, 2, 3, \ldots, k\}$. Under equilibrium conditions, the continuity equations connecting the probabilities introduced are:

$$-\lambda(0) p(0) + \sum_{s=1}^{k} \mu_s(1)\, p(1,s) = 0 \tag{1}$$

$$-\left[\lambda(1) + \mu_r(1)\right] p(1,r) + \sigma_r \lambda(0)\, p(0) + \sigma_r \sum_{s=1}^{k} \mu_s(2)\, p(2,s) = 0 \tag{2}$$

$$-\left[\lambda(n)+\mu_r(n)\right] p(n,r)+\lambda(n-1)\, p(n-1,r)+$$

$$\sigma_r \sum_{s=1}^{k} \mu_s(n+1)\, p(n+1,s)=0 \tag{3}$$

$$(n=2,3,\ldots,N-1)$$

$$-\mu_r(N)\, p(N,r)+\lambda(N-1)\, p(N-1,r)=0 \tag{4}$$

where equations (1), (2), (3) and (4) are valid for $r = 1,2,3,\ldots,k$.

In addition to the equations (1), (2), (3) and (4), we have also the following equation stating the normalizing condition

$$\sum_{n=1}^{N} \sum_{r=1}^{k} p(n,r) + p(0) = 1 \tag{5}$$

Our problem is to solve for the $Nk + 1$ probabilities connected with with $Nk + 2$ equations (1) through (5). Equations (1) through (4) are $Nk + 1$ homogeneous linear equations in as many unknowns and therefore we can find a non-trivial solution provided the determinant formed by the coefficients of the probabilities is zero. But equation (5) states that a non-trivial solution exists. Thus, we solve equations (1) through (4) in terms of any one of the probabilities, say $p(0)$, and then determine $p(0)$ uniquely by using (5). The solution procedure is more or less based on a paper by Ancker and Gafarian (1961).

Summing up equations (2) over r and using (1), we get

$$\lambda(1) \sum_{r=1}^{k} p(1,r) = \sum_{s=1}^{k} \mu_s(2)\, p(2,s) \tag{6}$$

Let us now prove the following result by induction

$$\lambda(m) \sum_{r=1}^{k} p(m,r) = \sum_{s=1}^{k} \mu_s(m+1)\, p(m+1,s) \tag{7}$$

$$(m = 1,2,\ldots,N-1)$$

Suppose (7) is true for $m = n - 1$, therefore

$$\lambda(n-1)\sum_{r=1}^{k} p(n-1,r) = \sum_{s=1}^{k} \mu_s(n)\, p(n,s) \tag{8}$$

Summing up equation (3) over r and using (8), we get (7) for $m = n$. Since the result (7) is true for $m = 1$, because of equation (6), we prove the validity of equation (7).

Equations (2), (3) on using (7) become

$$\lambda(n-1)\, p(n-1,r) = \left[\lambda(n) + \mu_r(n)\right] p(n,r) - \sigma_r \lambda(n) \sum_{s=1}^{k} p(n,s) \tag{9}$$

$$(n = 1,2,\ldots,N-1)$$

where for $n = 1$, left hand side equals $\sigma_r \lambda(0)\, p(0)$.

Equation (9) can be written in the matrix form

$$\underset{\sim}{A}\underset{\sim}{P} = \lambda(n-1)\,\underset{\sim}{Q} \tag{10}$$

where

$$\underset{\sim}{A} = \|a_{ij}\|$$

such that

$$a_{ij} = -\sigma_i \lambda(n) \qquad (i \neq j)$$

$$a_{ii} = \lambda(n) + \mu_i(n) - \sigma_i \lambda(n)$$

and $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are the column matrices

$$\left[p(n,1),\ p(n,2),\ldots\ldots\ldots,\ p(n,k)\right]$$

and

$$\left[p(n-1,1),\ p(n-1,2),\ldots\ldots\ldots,\ p(n-1,k)\right]$$

respectively.

We observe that $\underset{\sim}{A}^{-1}$ is given by

$$\underset{\sim}{A}^{-1} = \left\| a'_{ij} \right\|$$

such that

$$a'_{ij} = \frac{\sigma_i \, \lambda(n)}{B_n \, A_{n,i} \, A_{n,j}} \qquad (i \neq j)$$

$$a'_{ii} = \frac{B_n + \sigma_i / A_{n,i}}{B_n \, A_{n,i}}$$

where

$$A_{n,s} = \lambda(n) + \mu_s(n)$$

and

$$B_n = 1 - \lambda(n) \sum_{s=1}^{k} (\sigma_s / A_{n,s}) .$$

Substituting the value of $\underset{\sim}{A}^{-1}$ in (10), we get for $i = 1,2,\ldots,k$

$$p(1,i) = \frac{\lambda(0) \;\; p(0)\,\sigma_i}{A_{1,i} \displaystyle\sum_{s=1}^{k} \frac{\sigma_s \, \mu_s(1)}{A_{1,s}}} \qquad (11)$$

$$p(n,i) = \frac{\lambda(n-1)}{A_{n,i}} \left[p(n-1,i) + \frac{\lambda(n)\sigma_i \displaystyle\sum_{s=1}^{k} \frac{p(n-1,s)}{A_{n,s}}}{\displaystyle\sum_{s=1}^{k} \frac{\sigma_s \, \mu_s(n)}{A_{n,s}}} \right] \qquad (12)$$

$$(n = 2,3,\ldots,N-1)$$

Also from equation (4), we have

$$p(N,i) = \frac{\lambda(N-1) \, p(N-1,i)}{\mu_i(N)} \qquad (13)$$

$$(i = 1,2,\ldots,k)$$

The recurrence relations contained in equations (11), (12) and (13) are the required recurrence relations which may be used to compute the probabilities $p(n,i)$ in terms of $p(0)$, which may be uniquely evaluated by using equation (5).

We may also deduce from equations (11), and (12) the following relation giving $p(n,i)$ explicitly in terms of $p(0)$,

$$p(n,i) = p(0)\left[\prod_{i=0}^{n-1}\lambda(i)\right]\left(\prod_{p=1}^{n-1}\sum_{s_p=1}^{k}\left[\left\{\prod_{r=1}^{n-1}\frac{\delta_{s_r,s_{r-1}}+C_{n-r+1,s_{r-1}}}{A_{n-r+1,s_r}}\right\}\frac{\sigma_{s_{n-1}}}{A_{1,s_{n-1}}\sum_{s=1}^{k}\frac{\sigma_s \mu_s(1)}{A_{1,s}}}\right]\right)$$

$$(n = 2,3,\ldots,N-1;\; i = 1,2,\ldots,k)$$

where

$$C_{n,i} = \frac{\lambda(n)\,\sigma_i}{\sum_{s=1}^{k}\frac{\sigma_s\,\mu_s(n)}{A_{n,s}}},$$

$$s_0 = i,$$

$\prod_{p=1}^{n-1}\sum_{s_p=1}^{k}$ stands for $\sum_{s_1=1}^{k}\sum_{s_2=1}^{k}\cdots\sum_{s_{n-1}=1}^{k}$

and $p(0)$ may be evaluated by using equations (11), (13) and (5).

But, however, it must be admitted that for numerical calculations of $p(n,i)$ recurrence relations (11), (12), (13) and equation (5) are more suitable than equations (11), (13), (14), and (5). If, however, N is allowed to become infinite also, none of the forms of the solution given above help us find the value of $p(0)$ because to find the value of $p(0)$, we have to turncate the series somewhere which we cannot do if N is allowed to become infinite. Thus the solution sought above should be used only when N is finite. (See remarks after equation (31) also).

Mean Waiting Time

Let $w(t)$ be the probability density function of the waiting time distribution, then

$$w(t)\,dt = p(0)\,\delta(t)\,dt + \sum_{r=1}^{k}\sum_{n=1}^{N} p(n,r)\,\sigma_r\,\mu_r(n)\,e^{-\mu_r(n)\,t}\,\frac{[\mu_r(n)\,t]^{n-1}}{\lfloor n-1}\,dt \qquad (15)$$

where $\delta(t)$ is the Dirac-delta function.

Taking Laplace transform of both sides of (15) with parameter s, we get

$$\bar{w}(s) = p(0) + \sum_{r=1}^{k}\sum_{n=1}^{N} \sigma_r\,p(n,r)\left[\frac{\mu_r(n)}{s+\mu_r(n)}\right]^n \qquad (16)$$

$$\text{Mean waiting time} = -\frac{d}{ds}\,\bar{w}(s)\Big|_{s=0}$$

$$= \sum_{r=1}^{k}\sum_{n=1}^{N}\frac{n\,\sigma_r\,p(n,r)}{\mu_r(n)} \qquad (17)$$

In the next sub-section, we consider some particular forms of the state functions and draw some graphs showing the behaviour of some of the queue characteristics.

PARTICULAR CASES

(i) The Queuing System $M/E-M_k/1$

Substitute

$$\lambda(n) = \lambda\,, \quad \mu_r(n) = \mu_r$$

in (11), (12) and (13), i.e. the input distribution is Poisson with constant parameter λ and the service time distribution in the rth branch of the service channel is negative exponential with constant parameter μ_r.

This case has already been studied by Gupta and Goyal (1964a) by using the technique of generating functions. But the results obtained in (1964a) are not amenable to numerical calculations very far as are those obtained in the present study. This problem has already been deduced as a particular case in section I of chapter III.

(ii) Queues with Balking -- An application to 'Many Type Stoppage Machine Interference Problem'

Substitute

$$\lambda(n) = \frac{N-n}{N}\lambda \ , \quad \mu_s(n) = \mu_s$$

in (11), (12) and (13), so that $\lambda(n)$ depends linearly upon n. This case can be interpreted in two different ways as follows:

Firstly, suppose that the units arrive at a service facility following a Poisson process with parameter λ. Before joining the queue, the unit may decide not to join the queue at all, i.e. it may balk. When there are n units in the system, let the balking probability be n/N, then the effective input rate is as indicated by $\lambda(n)$ above.

Secondly, suppose that there are N machines under the supervision of one repairman. The machines fail at random according to a Poisson process with constant parameter λ/N and the machines are liable to k types of failures. The service time distribution for the sth type of repair is negative exponential with constant parameter μ_s. In such a situation also, the queue parameters are as indicated.

It may be pointed out here that Benson and Cox (1951) wrote the queue equations for two types of stoppages but could not give any solution as has also been remarked by Saaty (1961). Although the assumptions underlying

our case (ii) are different from those of Benson and Cox (1951), because we are not considering any priorities, but the solution of the problem obtained in case (ii) is essentially of a similar generalised problem as sought by Benson and Cox (1951). It may be remarked here that Jaiswal and Thiruvengadam (1963) have solved a more general problem than that posed by Benson and Cox (1951).

For the queuing system M/M/1, Ancker and Gafarian (1963) studied this type of arrival parameter. They also introduced one reneging parameter, α. However, if we also introduce a similar parameter, the solution cannot be obtained because we may observe then that equation (9) becomes a second order homogeneous linear difference equation with variable coefficients. <u>This problem is thus still open to be solved.</u>

(iii) <u>Queues with State Dependant Parameters</u>

Substitute

$$\lambda(n) = (n-1)^b \lambda \quad , \quad \mu_s(n) = (n+1)^c \mu_s$$

in equations (11), (12) and (13) where b and c are constants.

This type of state dependant parameters have been studied for the queuing system M/M/1 by Conway and Maxwell (1961).

Obviously when b = 0, c = 0, case (iii) reduces to case (i).

<u>Numerical Work</u>

Graphs showing the behaviour of the queue characteristics

(i) probability of no delay, i.e. $p(0)$,

(ii) mean number of units in the system, and

(iii) mean waiting time,

for all the three cases (i), (ii) and (iii), and

(iv) fraction of customers lost

for case (i) only for N = 5, 10, 15, 20

are drawn against λ/μ for N = 15, k = 5, $\mu_s = k\,\sigma_s\,\mu$, $\mu = 6$, $\sigma_1 = .10$, $\sigma_2 = .12$, $\sigma_3 = .20$, $\sigma_4 = .28$, $\sigma_5 = .30$ and various values of λ. In case (iii), graphs are drawn for (a) b = -1, c = -1, (b) b = 0, c = -1, (c) b = -1, c = 0, (d) b = 0, c = 0, i.e. case (i).

It is observed from numerical calculations that for case (i), for the same value of λ/μ, the values of p(0) and mean number of units in the system depend only upon the ratio λ/μ and not upon λ and μ separately whereas the fraction of customers lost and the mean waiting time depend upon λ and μ separately, inasmuchas doubling the value of μ, keeping λ fixed, means doubling the value of mean waiting time and of the fraction of customers lost.

Queuing System M/M/1 with State Dependant Parameters

Substituting

$$\sigma_r = \delta_{rs}, \quad \mu_s(n) = \mu(n), \quad \sum_{i=1}^{k} p(n,i) = p(n)$$

in equations (11), (12) and (13), we get

$$p(n) = \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}\, p(0) \qquad (n = 1,2,\ldots,N) \tag{18}$$

and using (5), we get

$$p(0) = \left[1 + \sum_{n=1}^{N}\left(\prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}\right)\right]^{-1} \tag{19}$$

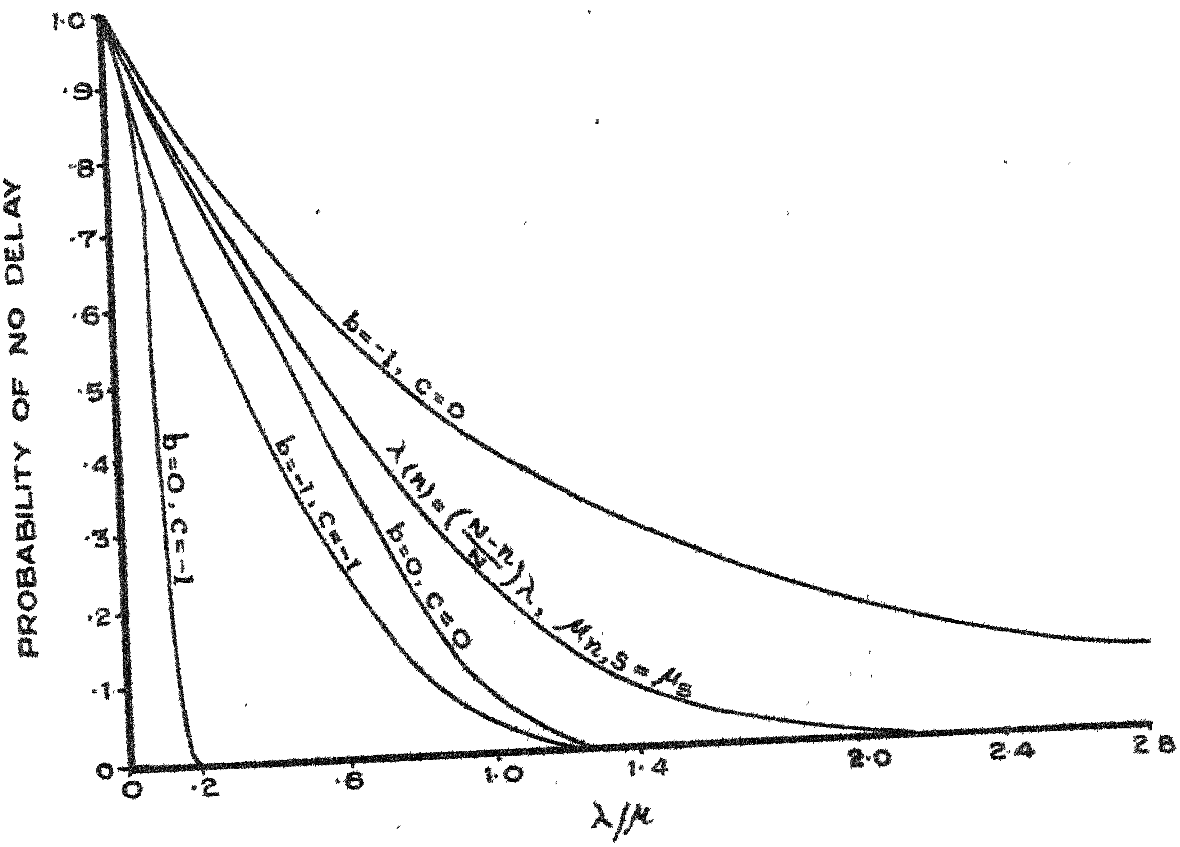
The results of all the papers cited in the first para of the introduction of this chapter may now be obtained from (18) and (19) by giving suitable forms to $\lambda(n)$ and $\mu(n)$. Thus by putting

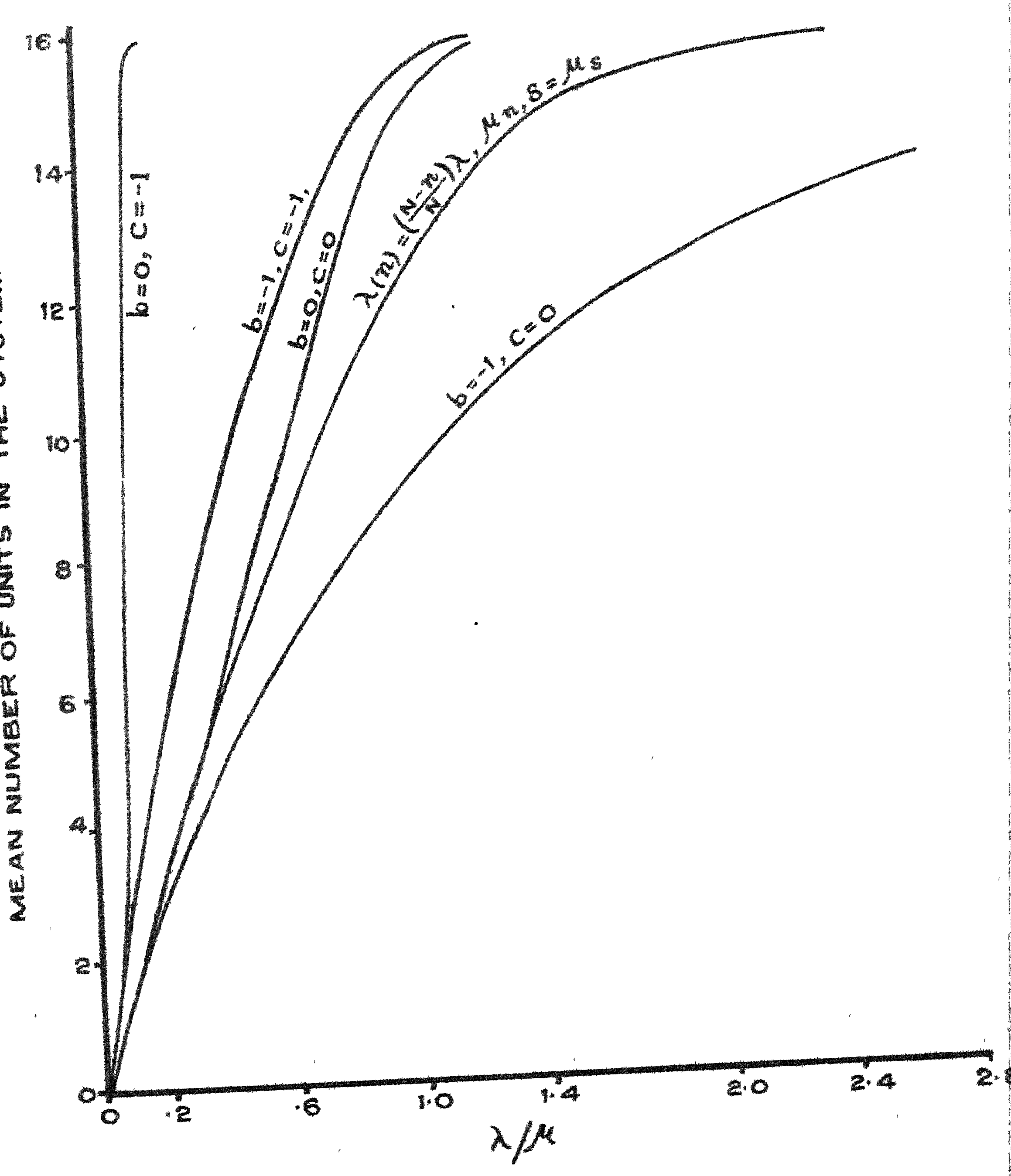
$$\text{(i)} \quad \lambda(n) = (1 - n/N)\lambda ,$$
$$\mu(n) = \mu + (n-1)\alpha$$

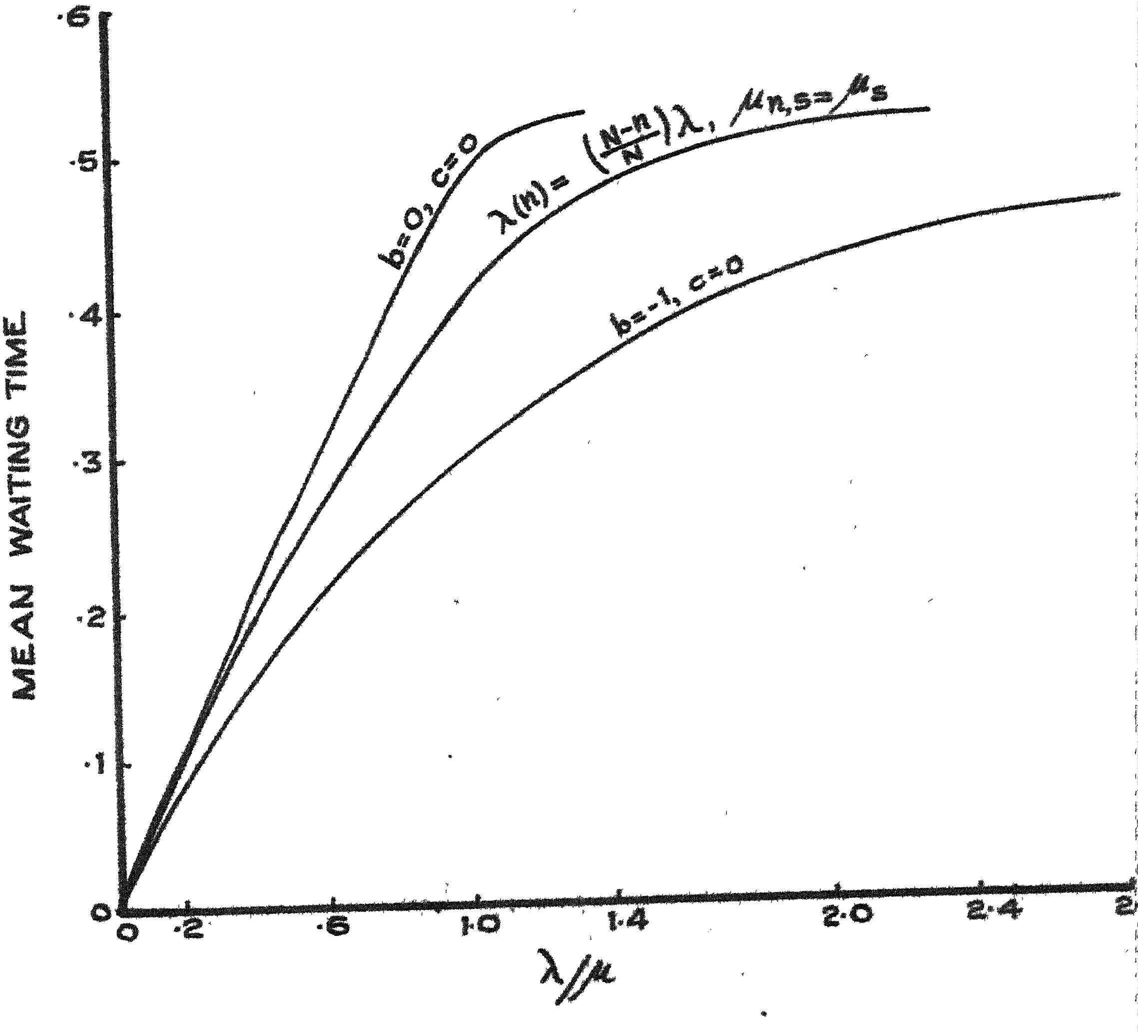
$$\text{(ii)} \quad \lambda(0) = 1, \quad \lambda(n) = \beta/n \text{ for } n = 1,2,\ldots,$$
$$\mu(n) = \mu + (n-1)\alpha$$

we get the results of Ancker and Gafarian (1963, 1963a).

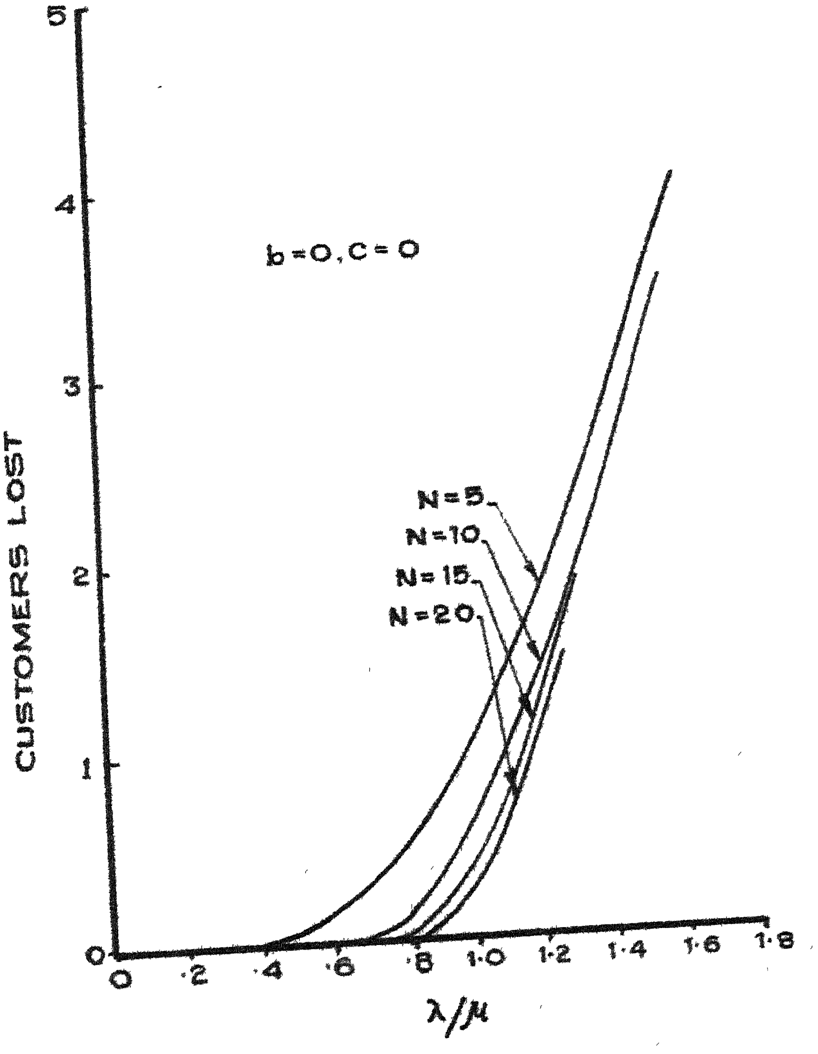
In fact, from equations (18) and (19) one may discuss the queuing system M/M/c where c may be finite or infinite by suitably choosing $\lambda(n)$ and $\mu(n)$, some of which are given in Saaty (1961).


PROBABILITY OF NO DELAY
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
O
·2
·6
1·0
1·4
2·0
2·4
2·8
λ/μ
b=-1, C=0
λ(n)=(N-n/N)λ, μn,s=μs
b=0, C=0
b=-1, C=-1
b=0, C=-1


MEAN NUMBER OF UNITS IN THE SYSTEM
16
14
12
10
8
6
4
2
0
0
·2
·6
1·0
1·4
2·0
2·4
λ/μ
b=0, C=-1
b=-1, C=-1,
b=0, C=0
λ(n)=((N-n)/N)λ, μn, S=μs
b=-1, C=0



MEAN WAITING TIME
λ/μ
b=0, c=0
λ(n) = (N-n / N)λ, μn,s = μs
b=-1, c=0
·6
·5
·4
·3
·2
·1
0
0
·2
·6
1·0
1·4
2·0
2·4



CUSTOMERS LOST
0
1
2
3
4
5
b = 0, C = 0
N = 5
N = 10
N = 15
N = 20
0
·2
·4
·6
·8
1·0
1·2
1·4
1·6
1·8
λ/μ

SECTION II: The Queuing System $H-M_k/M/1$ with state dependent parameters

The following problem is considered in this section:

Suppose that units arrive at a service facility which serves the units at random according to the negative exponential distribution with parameter $\mu(n)$ when there are n units in the system (queue + service), i.e. the first order probability that a unit is served in time dt is $\mu(n)$ dt when there are n units in the system. The arrival channel consists of k independent branches. A unit arriving for service enters the rth branch a fraction σ_r of the time on the average, so that $\sum_{r=1}^{k}\sigma_r = 1$. Only one unit can enter any branch at a time and if there is a unit present in any one of the k branches, no other unit can enter any other branch. We further assume that there is a reservoir of infinite capacity which emits a unit as soon as the arrival channel is free, so that the arrival channel is never free. The unit in the rth branch enters the system (queue or service) at the rate $\lambda_r(n)$ per unit time when there are n units already in the system, i.e., the first order probability that a unit enters the system from the rth branch of the arrival channel in time dt is $\lambda_r(n)$ dt when there are n units in the system. The queue discipline is assumed to be first come, first served. The maximum number of units in the system is allowed to be N, so that when a unit comes from the arrival channel and finds N units in the system, it is not allowed to join the queue and it overflows, i.e. is lost to the system.

Continuity Equations and Their Solution

Let p(n,r) denote the equilibrium probability that there are n units in the system, the unit in the arrival channel being in the rth branch, i.e.

the phase space consists of the elements of the Cartesian product $I \times J$ where I and J are the finite sets $I = \{0,1,2,\ldots,N\}$ and $J = \{1,2,\ldots,k\}$. Under equilibrium condition, the continuity equations connecting the probabilities introduced are:

$$\mu(1)\ p(1,r) - \lambda_r(0)\ p(0,r) = 0 \tag{20}$$

$$\sigma_r \sum_{s=1}^{k} \lambda_s(m-1)\ p(m-1,s) + \mu(m+1)\ p(m+1,r) - A_{m,r}\ p(m,r) = 0 \tag{21}$$

$$(m = 1,2,\ldots,N-1)$$

$$\sigma_r \sum_{s=1}^{k} \lambda_s(N-1)\ p(N-1,s) + \sigma_r \sum_{s=1}^{k} \lambda_s(N)\ p(N,s) - A_{N,r}\ p(N,r) = 0 \tag{22}$$

where

$$A_{j,r} = \mu(j) + \lambda_r(j), \qquad j = 1,2,\ldots,N$$

and all the above equations are valid for $r = 1,2,\ldots,k$.

In addition to equations (20), (21), (22) we have also the following equation stating the normalizing condition

$$\sum_{n=0}^{N} \sum_{s=1}^{k} p(n,s) = 1 \tag{23}$$

Our problem is to solve for the $(N+1)k$ probabilities connected with the $(N+1)k+1$ equations (20) through (23). Equations (20) through (22) are $(N+1)k$ homogeneous linear equations in as many unknowns and therefore we can find a non-trivial solution provided the determinant formed by the coefficients is zero. But equation (23) ensures that a non-trivial solution exists and hence the determinant formed by the coefficients is zero. Thus, we solve equations (20) through (22) in terms of any one of the probabilities, say $p(N,k)$, and then determine $p(N,k)$ uniquely from (23).

Summing up equation (21) over r, and adding the result obtained for $m = 1,2,\ldots,n$ say, $(n = 1,2,\ldots,N-1)$, we get

$$\mu(n+1)\sum_{s=1}^{k} p(n+1,s) - \sum_{s=1}^{k}\lambda_s(n)\, p(n,s) = \mu(1)\sum_{s=1}^{k} p(1,s) - \sum_{s=1}^{k}\lambda_s(0)\, p(0,s)$$

$$= 0, \text{ using (20)} \tag{24}$$

Using the value of $\sum_{s=1}^{k}\lambda_s(m-1)\, p(m-1,s)$ from (24) in (21), we get

$$A_{m,r}\, p(m,r) - \sigma_r\, \mu(m)\sum_{s=1}^{k} p(m,s) = \mu(m+1)\, p(m+1,r) \tag{25}$$

which can be written in the matrix form as

$$\underset{\sim}{B}\underset{\sim}{P} = \mu(m+1)\, \underset{\sim}{Q} \tag{26}$$

where $\underset{\sim}{B}$ is the matrix

$$\underset{\sim}{B} = \|b_{ij}\|$$

such that

$$b_{ij} = -\sigma_i\, \mu(m) \qquad (i \neq j)$$

$$b_{ii} = A_{m,i} - \sigma_i\, \mu(m)$$

and $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are the column matrices

$$\left[p(m,1),\ p(m,2),\ldots\ldots\ldots\ldots,\ p(m,k)\right]$$

and

$$\left[p(m+1,1),\ p(m+1,2),\ldots\ldots\ldots\ldots,\ p(m+1,k)\right]$$

respectively.

Now $\underset{\sim}{B}^{-1}$ is given by

$$\underset{\sim}{B}^{-1} = \|b'_{ij}\|$$

such that

$$b'_{ij} = \frac{\mu(m)\,\sigma_i}{A_{m,i}\,A_{m,j}\,D_m} \qquad (i \neq j)$$

$$b'_{ii} = \frac{D_m + \mu(m)\,\sigma_i/A_{m,i}}{A_{m,i}\,D_m}$$

where

$$D_m = 1 - \mu(m) \sum_{s=1}^{k} (\sigma_s/A_{m,s}) .$$

Using this value of $\underset{\sim}{B}^{-1}$ in (26), we get

$$p(m,r) = \frac{\mu(m+1)}{A_{m,r}} \left[p(m+1,r) + \frac{\mu(m)\,\sigma_r \sum_{s=1}^{k} p(m+1,s)/A_{m,s}}{\sum_{s=1}^{k} \sigma_s\,\lambda_s(m)/A_{m,s}} \right] \qquad (27)$$

$$(m = 1,2,\ldots,N-1;\; r = 1,2,\ldots,k)$$

Similarly using the value of $\sum_{s=1}^{k} \lambda_s(N-1)\, p(N-1,s)$ from (24) in (22), one gets

$$\sigma_r \sum_{s=1}^{k} A_{N,s}\, p(N,s) - A_{N,r}\, p(N,r) = 0 \qquad (28)$$

$$(r = 1,2,\ldots,k)$$

It is easy to verify that the determinant formed by the coefficients of $p(N,r)$, $r = 1,2,\ldots,k$, is zero and therefore we can solve equation (28) for any $k - 1$ probabilities involved in terms of the kth. Let us solve for $p(N,i)$, $i = 1,2,\ldots,k-1$, in terms of $p(N,k)$. Leaving out the kth equation, we have the matrix representation of (28) as

$$\underset{\sim}{C}\,\underset{\sim}{P} = - A_{N,k}\, p(N,k)\, \underset{\sim}{S} \tag{29}$$

where $\underset{\sim}{C}$ is the $(k-1) \times (k-1)$ matrix

$$\underset{\sim}{C} = \| c_{ij} \|$$

such that

$$c_{ij} = \sigma_i A_{N,j} \qquad (i \neq j)$$

$$c_{ii} = (\sigma_i - 1) A_{N,i}$$

and $\underset{\sim}{P}$ and $\underset{\sim}{S}$ are the column matrices

$$\left[p(N,1),\ p(N,2), \ldots\ldots,\ p(N,k-1) \right]$$

and

$$\left[\sigma_1,\ \sigma_2, \ldots\ldots\ldots\ldots, \sigma_{k-1} \right]$$

respectively.

Now, $\underset{\sim}{C}^{-1}$ is given by

$$\underset{\sim}{C}^{-1} = \| c'_{ij} \|$$

such that

$$c'_{ij} = - \frac{\sigma_i}{A_{N,i}\, \sigma_k} \qquad (i \neq j)$$

$$c'_{ii} = - \frac{\sigma_i + \sigma_k}{A_{N,i}\, \sigma_k}$$

Substituting this value of $\underset{\sim}{C}^{-1}$ in (29), we get

$$p(N,i) = \frac{\sigma_i A_{N,k}\, p(N,k)}{\sigma_k A_{N,i}} \qquad (i = 1,2,\ldots,k-1) \tag{30}$$

Also, equation (20) gives

$$p(0,r) = \frac{\mu_{(1)}\, p(1,r)}{\lambda_r(0)}, \qquad (31)$$

$$(r = 1,2,\ldots,k)$$

Equations (27), (30), (31) are the required recurrence relations which give all the probabilities in terms of $p(N,k)$, which itself may now be determined by using the normalizing equation (23) and hence all the probabilities are completely known in terms of the queue parameters.

It is quite obvious from the solution sought above (and also in the previous section of this chapter) that the solutions are computer oriented solutions. It may also be noted that if N is allowed to become infinite, we may not be able to find the value of $p(N,k)$ because of the complexity of recurrence relation (27) and the inability of the computer to recognise infinity. The only possibility of solving the problem without a bound on N is then to see when the probabilities become arbitrarily small and fix a bound there for the calculation of $p(N,k)$. But this is a tedious procedure because for a different set of queue parameters we will have to fix up different bounds. Thus it is in the case when there is no bound on N that the method outlined in this chapter may not be very suggestive and it is in such cases that we see the superiority of the method of generating functions where no such difficulty arises even in numerical calculations, although the calculations involved are prohibitive, particularly when the number of branches in the arrival channel (or service channel) is large. Of course, if the arrival and/or service parameters are state dependent, the method of

generating functions cannot probably be applied with success even if we specify some simple particular forms of the functions defining the dependance of the queue parameters upon the state of the system. Moreover, the method of generating functions cannot be used with ease to yield explicitly all the probabilities involved, particularly when N is finite and the problem under consideration is complicated, as here. Considering Bellman and Brock's (1959) concept of the solution of a problem, we can safely say that the method of generating function does not yield a solution, particularly when the problem being solved is complicated.

In the next section, we specify some particular forms of the state functions and draw some graphs showing the behaviour of some of the queue characteristics.

PARTICULAR CASES

(i) Multiple Server Queues With Hyper-Poisson Input and Exponential Service Time Distribution

Substitute

$$\lambda_r(n) = \lambda_r,$$

$$\mu(n) = n\mu, \qquad \text{for } n < s$$

$$= s\mu, \qquad \text{for } n \geq s$$

in equations (27), (30) and (33) so that one gets the recurrence relations for the s server queues with hyper-Poisson input and exponential service time distribution.

Nishida (1962) has recently considered such a queuing system when $k = 2$ and $\lambda_r = k\,\sigma_r\,\lambda$.

Thus the solution of the problem sought in case (i), is a generalisation of Nishida's (1962) paper in the same sense as the generalisation of Morse's (1958) work considered by Gupta and Goyal (1964b).

(ii) Queues with Reneging

Substitute

$$\lambda_r(n) = \lambda_r,$$

$$\mu(n) = \mu + (n-1)\alpha$$

where α is a constant, in equations (27), (30) and (31).

This type of α in the service parameter may be interpreted as the reneging parameter, i.e. the parameter which accounts for the units leaving the queue without being served. Suppose that each incoming unit has a fixed time in its mind to wait in the queue. If its service does not begin by that time, it reneges, i.e. it leaves the queue after joining the queue and before getting into the service facility. This fixed time, we assume is a random variable following the Poisson distribution with parameter α. Then the parameter for usual service plus those reneging is as indicated above.

This type of reneging parameter has been studied for the queuing system M/M/1 by Ancker and Gafarian (1963, 1963a).

(iii) Queues with Partial Reneging

Suppose that the units are not allowed to renege (or the units do not renege of their own) till a fixed queue length is reached. In such a case, with the type of reneging described above, we may have the following type of queue parameters

$$\lambda_r(n) = \lambda_r,$$

$$\mu(n) = \mu \quad , \text{ for } \quad n < m$$

$$= \mu + (n-1)\alpha, \quad n \geqslant m.$$

(iv) Queues with State Dependant Parameters

Substitute

$$\lambda_r(n) = (n+1)^b \lambda_r,$$

$$\mu(n) = n^c \mu$$

in equations (27), (30) and (31) where b and c are constants. One may now give various values to b and c , positive or negative, and study the queues in which the mean arrival and/or service rate depends upon the state of the system, linearly or non-linearly.

For example, if

$$b = c = 0$$

then one studies the queues with hyper -Poisson input and exponential service time distribution. This case has already been studied by Gupta and Goyal (1964b) by using the technique of generating functions and has been reported in section II of chapter III of this thesis. But the results obtained in (1964b) are not amenable to numerical calculations very far as are those obtained in the present study.

(v) Queuing System M/M/1 with State Dependant Parameters

By substituting

$$\sigma_r = \delta_{rs}, \quad \lambda_r(n) = \lambda(n),$$

$$\sum_{r=1}^{k} p(n,r) = p(n),$$

in equations (27), (30) and (31), one may once again verify the equations (19) and (18) of this chapter.

NUMERICAL WORK

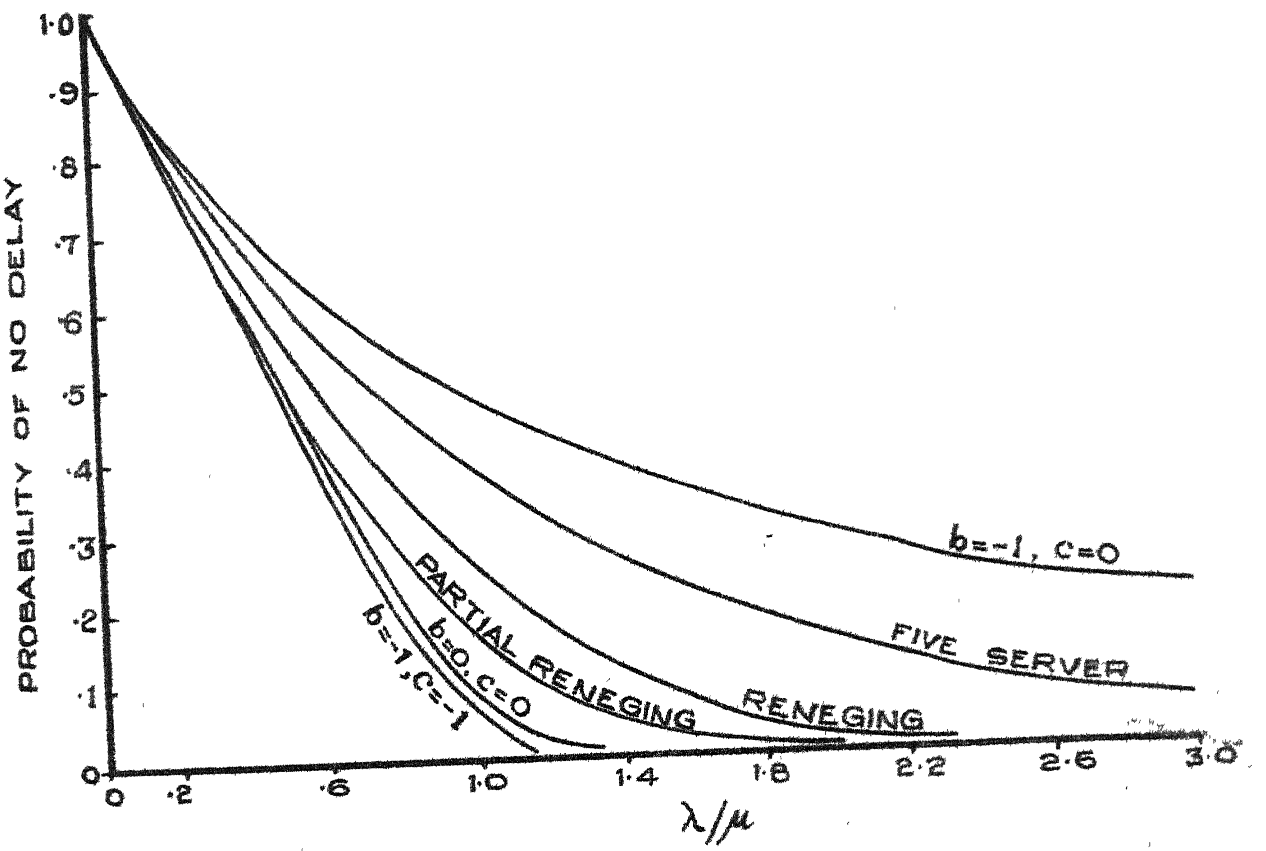
Graphs showing the behaviour of the queue characteristics

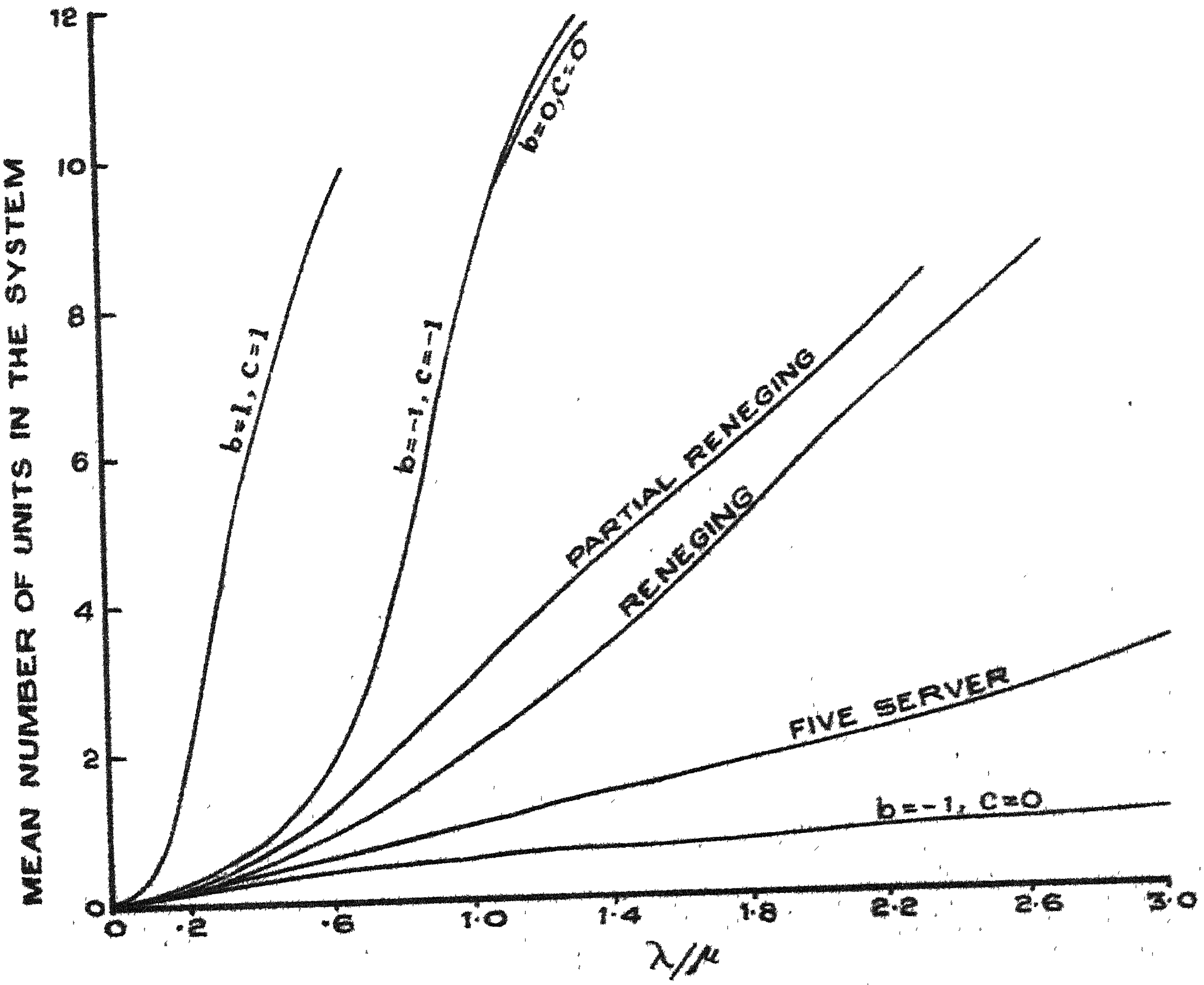
(i) probability of no delay, i.e. $\sum_{r=1}^{k} p(0,r)$,

(ii) mean number of units in the system, i.e. $\sum_{n=0}^{N}\sum_{r=1}^{k} n\, p(n,r)$

are drawn against λ/μ for $N = 15$, $k = 5$, $\lambda_r = k\sigma_r\lambda$, $\mu = 6$, $\sigma_1 = .10$, $\sigma_2 = .12$, $\sigma_3 = .20$, $\sigma_4 = .28$, $\sigma_5 = .30$ and various values of λ.

In case (i), the graphs are drawn when $s = 5$; in case (ii) $\alpha = 1.2$, in case (iii) $m = 6$, $\alpha = 1.2$ and in case (iv) for (a) $b = -1$, $c = -1$, (b) $b = -1$, $c = 0$, (c) $b = 0$, $c = 0$, and (d) $b = 1$, $c = 1$.


PROBABILITY OF NO DELAY
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
0
·2
·6
1·0
1·4
1·8
2·2
2·6
3·0
λ/μ
b=-1, c=0
FIVE SERVER
RENEGING
PARTIAL RENEGING
b=0, c=0
b=-1, c=-1


MEAN NUMBER OF UNITS IN THE SYSTEM
12
10
8
6
4
2
0
b=1, C=1
b=-1, C=-1
b=0, C=0
PARTIAL RENEGING
RENEGING
FIVE SERVER
b=-1, C=0
0
·2
·6
1·0
1·4
1·8
2·2
2·6
3·0
λ/μ

CHAPTER V

BULK QUEUES

Thus far we were concerned with queues in which units arrive and/or are served singly. However, a well-developed theory of bulk queues exists today. Reference to this effect may be made to: Foster (1961, 1964), Conolly (1960), Jaiswal (1961, 1960a, 1960b), Natrajan (1962), Foster and Nyunt (1961), Foster and Perera (1964), Bhat (1964), Keilson (1962), Miller (1959), Bailey (1954), Downton (1955), Arora (1964a) etc.

All these authors except Keilson (1962) have assumed that the size of the batch is fixed. But we are all familiar with situations in which the batch, particularly the arriving batch, is not fixed. Thus Gupta and Goyal (1964c) considered a queuing system with batch Poisson arrivals and hyper-exponential service time distribution, the arriving batch size being variable. Again Gupta (1964) considered the queuing system with batch Poisson arrival and a general class of service time distributions, the arriving batch size being variable. Later, however, these two queuing systems were combined together by Gupta (1965a) and studied as a particular case of the problem considered in section I of chapter III. This generalised system is reproduced here in section I and the two particular systems, referred to above, are studied in details.

Moreover, all the authors mentioned above have assumed that the mean arrival and service rates are constant. However, the author found it possible to relax this assumption for the queuing system $M\text{-}H_k/M/1$ when the service is in batches of fixed size S, or the whole queue length, whichever

is less. To be more precise, it is possible to solve the problem considered in section II of chapter IV when service is in batches of fixed size S, or the whole queue length, whichever is less. This problem is reproduced here in section II.

SECTION I: The Queuing System M/M IM-$E_{nk}/1$ with Batch Arrivals

In the problem to be considered in this section, we assume that the service time distribution and the queue discipline are the same as in the problem considered in section I of chapter III, but the arrival pattern changes. Suppose that arrivals again follow a Poisson distribution, but this time we allow batches of arriving units. Thus we assume that the probability of a batch arriving in time dt is λdt. Also let c_m be the probability that the arriving batch consists of m units, so that the probability of a batch consisting of m units to arrive in time dt is $\lambda c_m dt$. Assuming the waiting space for an infinite queue, the continuity equations for this queuing system are:

$$\frac{d}{dt} p(0,t) = -\lambda p(0,t) + \sum_{i=1}^{n} \mu_i p(1,1,i,t) \tag{1}$$

$$\frac{d}{dt} p(m,s,r,t) = -(\lambda + \mu_r) p(m,s,r,t) + d_s \sigma_r \sum_{i=1}^{n} \mu_i p(m+1,1,i,t)$$

$$+ \mu_r p(m,s+1,r,t) + \lambda \sum_{\ell=1}^{m-1} c_\ell p(m-\ell,s,r,t)$$

$$+ \lambda c_m \sigma_r d_s p(0,t) \tag{2}$$

$$(m = 1,2,3,\ldots\ldots)$$

where the notations used are the same as in section I of chapter III.

Now, proceeding exactly as we did in section I of chapter III, we observe that equation (14) of that section gives the Laplace transform of the system size distribution of the model under consideration, if in that equation we substitute

$$X_r = \frac{\mu_r}{u + \lambda - \lambda \sum_{\ell=1}^{\infty} c_\ell w^\ell + \mu_r}, \tag{3}$$

$$P_r = 0, \tag{4}$$

and

$$Q_r = \sigma_r D(X_r)\, \overline{p}(0,u)\left[u + \lambda - \lambda \sum_{\ell=1}^{\infty} c_\ell w^\ell\right] - \sigma_r D(X_r) w^r \tag{5}$$

Also the substitutions

$$X_r = \frac{\mu_r}{\lambda(1 - \sum_{\ell=1}^{\infty} c_\ell \theta_r^\ell) + \mu_r}, \tag{6}$$

$$P_r = 0, \tag{7}$$

and

$$Q_r = \lambda \sigma_r D(X_r)\, p(0)\left[1 - \sum_{\ell=1}^{\infty} c_\ell \theta_r^\ell\right] \tag{8}$$

in equation (25) of section I of chapter III yields the Laplace transform of the waiting time distribution in the steady state case of the problem under consideration.

However, we are not giving the final simplified results in the general case. In the particular cases when the service time distribution is (i) hyper-exponential, or (ii) mixed-Erlangian of the first kind, the results are obtained below.

The Queuing System $M/H\text{-}M_r/1$ with Batch Arrivals

Let $p(s,m,t)$ denote the probability that at time t there are m units in the system, the unit being served being in the sth branch of the service channel. Also let $p(0,t)$ be the probability of there being no unit in the system at time t. The continuity equations in this case are:

$$\frac{d}{dt}\, p(0,t) = -\lambda\, p(0,t) + \sum_{r=1}^{n} \mu_r\, p(r,1,t) \tag{9}$$

$$\frac{d}{dt}\, p(s,m,t) = -(\lambda + \mu_s)\, p(s,m,t) + \sigma_s \sum_{r=1}^{n} \mu_r\, p(r,m+1,t) + \lambda \sum_{j=1}^{m-1} c_j\, p(s,m-j,t) + \sigma_s \lambda\, c_m\, p(0,t) \tag{10}$$

$$(m = 1,2,3,\ldots;\; s = 1,2,\ldots,n)$$

Let us define the generating functions

$$G(w,r,t) = \sum_{m=1}^{\infty} p(s,m,t)\, w^m \tag{11}$$

Multiplying equation (10) by w^m, summing over m, using (9) and applying the Laplace transformation with parameter u, we get

$$\underset{\sim}{A}\underset{\sim}{G} = \left[w^P - \bar{p}(0,u)\Big(u + \lambda - \lambda \sum_{j=1}^{\infty} c_j w^j\Big) \right] \underset{\sim}{R} \tag{12}$$

where $\underset{\sim}{A}$ is the matrix given by

$$\underset{\sim}{A} = \| a_{ij} \|$$

such that

$$a_{kj} = -\sigma_k\, \mu_j / w, \qquad (k \neq j)$$

$$a_{kk} = u + \lambda - \lambda \sum_{j=1}^{\infty} c_j w^j + \mu_k - \sigma_k\, \mu_k / w,$$

$\underset{\sim}{G}$ and $\underset{\sim}{R}$ are respectively the column matrices,

$$\left[\bar{G}(w,1,u),\ \bar{G}(w,2,u),\ \ldots\ldots,\ \bar{G}(w,n,u)\right]$$

and

$$\left[\sigma_1,\ \sigma_2,\ \sigma_3,\ldots\ldots,\ \sigma_n\right]$$

and $p\ (>0)$ are the initial number of units with which the system starts at time $t = 0$.

We observe that $\underset{\sim}{A}^{-1}$ is given by

$$\underset{\sim}{A}^{-1} = \left\| b_{ij} \right\|$$

such that

$$b_{ij} = \frac{\sigma_i \, k_j}{B_i B_j \, X}, \qquad (i \neq j)$$

$$b_{ii} = \frac{X + \sigma_i \, k_i / B_i}{B_i \, X}$$

where

$$B_r = u + \lambda\left(1 - \sum_{j=1}^{\infty} c_j w^j\right) + k_r$$

and

$$X = w - \sum_{r=1}^{n} \frac{\sigma_r \, k_r}{B_r}.$$

Hence from (12), we get by substituting the expression for $\underset{\sim}{A}^{-1}$

$$\bar{G}(w,r,u) = \frac{w\,\sigma_r}{B_r\,X}\left[w^p - \bar{p}(0,u)\left(u + \lambda - \lambda\sum_{j=1}^{\infty} c_j w^j\right)\right] \qquad (13)$$

Thus $\bar{G}(w,r,u)$ have been expressed in terms of only one unknown, namely, $\bar{p}(0,u)$, which is determined by the analyticity conditions of $\bar{G}(w,r,u)$.

By Rouche's Theorem, it can be easily proved that the common denominator,

$$w \prod_{r=1}^{n} B_r - \sum_{r=1}^{n} \left[\sigma_r \mu_r (\prod_{\substack{i=1 \\ i \neq r}}^{n} B_i) \right]$$

of all $\bar{G}(w,r,u)$ has only one root inside the unit circle $|w|<1$. Thus if $\bar{G}(w,r,u)$ are to be analytic, the numerators must cancel this particular zero of the denominators. Using this fact, we get

$$\bar{p}(0,u) = \frac{w_o^{\,p}}{u + \lambda - \lambda \sum_{j=1}^{\infty} c_j w_o^{\,j}} \qquad (14)$$

where w_o is that root of the common denominator which lies inside the unit circle excluding $w = 0$.

Let us now define another generating function

$$\bar{G}(w,u) = \sum_{r=1}^{n} \bar{G}(w,r,u) \qquad (15)$$

and thus using (13) on the right hand side of (15), we get

$$\bar{G}(w,u) = \frac{\sum_{r=1}^{n} (\sigma_r / B_r)}{X} \left[w^{p+1} - w\, \bar{p}(0,u)(u + \lambda - \lambda \sum_{j=1}^{\infty} c_j w^j) \right] \qquad (16)$$

This is the required Laplace transform of the generating function from which the queue parameters are usually evaluated.

In case the initial number, p, of units obey a certain probability distribution, say with density function f(p), then

$$\bar{G}(w,u) = \sum_{p=1}^{N} \bar{G}(w,u)\; f(p) \qquad (17)$$

is the required Laplace transform of the generating function, N being the maximum number of initial units permissible.

We may now verify some earlier results from the results arrived at in this subsection. For instance:

(a) Substitute

$$\sigma_r = 1/n, \quad \mu_r = \mu$$

in (16), then

$$\overline{G}(w,u) + \overline{p}(0,u) = \frac{w^{p+1} + \mu(w-1)\overline{p}(0,u)}{w(u+\lambda+\mu) - \mu - \lambda w \sum_{j=1}^{\infty} c_j w^j} \tag{18}$$

which is the same as equation (6) of Luchak (1958).

(b) Substitute

$$c_j = \delta_{jk}$$

so that the batches become of equal size, k.

In particular, when k = 1, the batches become of equal size 1, i.e. the arrival time distribution is Poisson. Substituting these values of c_j in (16), we get

$$\overline{G}(w,u) = \frac{\left[w^p - \overline{p}(0,u)(u+\lambda-\lambda w)\right]\sum_{r=1}^{n}(\sigma_r/c_r)}{w - \sum_{r=1}^{n}(\sigma_r \mu_r/c_r)} \tag{19}$$

where now

$$c_r = u + \lambda(1-w) + \mu_r.$$

This in fact, is the generalisation of Gupta and Goyal (1964a), in the sense that in that paper we had assumed steady state conditions to obtain and (19) represents the Laplace transform of the system size distribution in the transient case of the same problem when an infinite waiting space is available.

(c) **Steady State Solution**

Assuming

$$\rho = \lambda\left[\sum_{r=1}^{n}(\sigma_r/k_r)\right]\left[\sum_{j=1}^{\infty}(j\,c_j)\right] < 1,$$

steady state solution is possible, so that by applying the Abel's Corollary, we get from (13) and (16)

$$G(w,s) = \frac{-w\,\sigma_s\,\lambda\,p(0)(1-\sum_{j=1}^{\infty}c_jw^j)}{D_s\left[w-\sum_{r=1}^{n}(\sigma_r k_r/D_r)\right]} \qquad (20)$$

$$G(w) = \frac{-\lambda w\,p(0)\,(1-\sum_{j=1}^{\infty}c_jw^j)\,\sum_{r=1}^{n}(\sigma_r/D_r)}{w-\sum_{r=1}^{n}(\sigma_r k_r/D_r)} \qquad (21)$$

where now

$$D_r = \lambda(1-\sum_{j=1}^{\infty}c_jw^j) + k_r.$$

Also since,

$$\lim_{w\to 1} G(w) + p(0) = 1 \qquad (22)$$

it being the normalising equation, we get

$$p(0) = 1 - \rho. \qquad (23)$$

Thus, by suitably defining the traffic intensity factor, ρ, we obtain the same value of $p(0)$, the probability of no delay, in the case of batch arrivals as in the case of single arrivals.

Putting

$$c_j = \delta_{j1} \text{ and } k_r = n\,\sigma_r\,k$$

in (21) and (23), we get

$$G(w) = \left(\frac{1}{1 + \rho \sum_{r=1}^{n} (\sigma_r / E_r)} - 1 \right) w\, p(0) \qquad (24)$$

and

$$p(0) = 1 - \rho , \quad (\rho = \lambda / \mu) \qquad (25)$$

where

$$E_r = \rho (w - 1) - n\sigma_r ,$$

the same as obtained earlier by Gupta and Goyal (1964a).

Waiting Time Distribution

Proceeding exactly as in section I of chapter III, we observe that the Laplace transform (with parameter v) of the waiting time distribution in the steady state case is given by

$$W(v) = p(0) + \sum_{r=1}^{n} G\left(\frac{\mu_r}{v + \mu_r} , r \right) \qquad (26)$$

where $G(\frac{\mu_r}{v + \mu_r}, r)$ may be obtained from (20) by substituting $w = \frac{\mu_r}{v + \mu_r}$.

The Queuing System M/1M-E_k/1 with Batch Arrivals

Let $p(s,m,t)$ denote the probability that at time t there are m units in the system (queue + service), the unit in service being in the sth phase. Also let $p(0,t)$ denote the probability of there being no unit in the system at time t. Observing that $p(s,0,t) = 0$, it being an impossible event, the continuity equations are:

$$\frac{d}{dt}\, p(s,m,t) = -(\lambda + \mu)\, p(s,m,t) + \mu\, p(s+1,m,t) +$$

$$\lambda \sum_{j=1}^{m-1} c_j\, p(s,m-j,t) + \mu\, d_s\, p(1,m+1,t) +$$

$$\lambda\, c_m\, d_s\, p(0,t) \qquad (27)$$

$$(m = 1,2,3,\ldots;\ s = 1,2,\ldots,k)$$

and

$$\frac{d}{dt}\, p(0,t) = -\lambda\, p(0,t) + \mu\, p(1,1,t) \qquad (28)$$

Let us define the generating function

$$G(z,w,t) = \sum_{m=1}^{\infty} \sum_{s=1}^{k} p(s,m,t)\, z^s w^m \qquad (29)$$

Multiplying equation (27) by $z^s w^m$, summing over s (1 to k) and m (1 to ∞), using (28), (29) and applying the Laplace transform with parameter u, we get

$$\left[u + \lambda + \mu - \mu/z - \lambda C(w)\right] \bar{G}(z,w,u)$$

$$= \lambda D(z)\, C(w)\, \bar{p}(0,u) + D(z)\, w^p - D(z)\, (u + \lambda)\, \bar{p}(0,u) -$$

$$\mu \left\{1 - D(z)\, w^{-1}\right\} \sum_{m=1}^{\infty} p(1,m,u)\, w^m \qquad (30)$$

where

$$D(z) = \sum_{s=1}^{k} d_s z^s,$$

$$C(w) = \sum_{m=1}^{\infty} c_m w^m,$$

and p (>0) is the initial number of units with which the system starts at time t $= 0$.

Substituting

$$s = \frac{\mu}{u + \lambda + \mu - \lambda G(w)} = x \text{ (say)}$$

in (30), we get

$$H(w) = \sum_{m=1}^{\infty} \overline{p}(1,m,u)\, w^m = \frac{w\,\overline{p}(0,u)\, D(x) \left[\lambda G(w) - (u + \lambda) \right] + D(x)\, w^{p+1}}{\mu \left[w - D(x) \right]} \qquad (31)$$

Now, it can easily be proved by Rouche's Theorem that there is only one zero inside the unit circle of the denominator of $H(w)$. In order that the infinite series, $H(w)$, be convergent atleast inside the unit circle, the numerator of $H(w)$ must cancel this zero of the denominator. Applying this condition, we obtain

$$\overline{p}(0,u) = \frac{w_0^{\,p}}{u + \lambda\left[1 - G(w_0)\right]} \qquad (32)$$

where w_0 is that root of the denominator of $H(w)$ which lies inside the unit circle.

Thus, (31) and (32) together represent the Laplace transform of the queue size distribution.

Mean System Length

The Laplace transform of the mean system length can be easily calculated from $\overline{G}(z,w,u)$ itself. We have

$$\text{Laplace transform of the Mean System Length} = \left. \frac{d}{dw}\overline{G}(1,w,u) \right|_{w=1}$$

$$= \frac{1}{u}\left[p + \mu \sum_{m=1}^{\infty} \bar{p}(1,m,u)\right] + \frac{\lambda}{u^2} \sum_{j=1}^{\infty} j\, c_j \tag{33}$$

where

$$\sum_{m=1}^{\infty} \bar{p}(1,m,u) = H(1) = \frac{\left[1 - u\,\bar{p}(0,u)\right]\, D(Y)}{\mu\left[1 - D(Y)\right]}$$

where

$$Y = \frac{\mu}{u+\mu}.$$

PARTICULAR CASES

(i) Exponential Service

Substituting

$$d_r = \delta_{r1}$$

where δ_{r1} is the Kronecker delta, we get

$$\bar{G}(1,w,u) + \bar{p}(0,u) = \frac{w^{p+1} + \mu\,\bar{p}(0,u)\,(w-1)}{w\left[u + \lambda + \mu - \lambda C(w)\right] - \mu} \tag{34}$$

a result due to Luchak (1958), equation (6).

(ii) Batches of Equal Numbers

If

$$c_k = \delta_{jk},$$

the arriving batches become of equal size j.

In particular, when $j = 1$, each arriving batch consists of only one unit. Then

$$\bar{G}(1,w,u) = \frac{1}{u + \lambda(1-w)}\Big[\lambda\bar{p}(0,u)\,w - w^p - \mu(1-w^{-1})\,H(w) - (u+\lambda)\,\bar{p}(0,u)\Big] \tag{35}$$

where

$$H(w) = \frac{\overline{p}(0,u)\ D(x)\ \left[\lambda (w-1) - u\right] + D(x)\ w^{p}}{\mu \left[1 - D(x)\ w^{-1}\right]} .$$

This is the time dependant solution of the problem considered by Jain (1962) when an infinite waiting space is allowed.

(iii) Steady State Solution

Applying the Abel's corollary to equation (30), we have

$$G(z,w) = \frac{1}{\lambda + \mu - \mu/z - \lambda C(w)} \Big[\lambda D(z)\ C(w)\ p(0) - \lambda D(z)\ p(0) - \mu \{1 - D(z)\ w^{-1}\} \sum_{m=1}^{\infty} p(1,m)\ w^{m} \Big] \tag{36}$$

where

$$\sum_{m=1}^{\infty} p(1,m)\ w^{m} = \frac{\lambda\ w\ p(0)\ D(x)\ \left[C(w) - 1\right]}{\mu \left[w - D(x)\right]}$$

and

$$x = \frac{\mu}{\lambda + \mu - \lambda C(w)} .$$

Waiting Time Distribution

Proceeding exactly as in section I of chapter III, we find that the Laplace transform (with parameter v) of the waiting time distribution in the steady state case is given by

$$W(v) = p(0) + \frac{G\left[\theta, D(\theta)\right]}{D(\theta)} \tag{37}$$

where

$$\theta = \frac{\mu}{\mu + v}$$

and $G\left[\theta, D(\theta)\right]$ may be obtained from equation (36). Thus we have

$$\bar{w}(v) = \frac{-v\, p(0)}{\lambda - v - \lambda\, c\left[p(z)\right]} \tag{38}$$

where

$$z = \frac{\mu}{\mu + v}.$$

SECTION II: The Bulk Service Queuing System M-M_k/M/1 with State Dependent Parameters

The problem considered in this section is a generalization of the problem considered in section II of chapter IV. Here we suppose that the units are served in batches of fixed size, S, or the whole queue length, whichever is less. In particular when S = 1, this problem reduces to the problem already reproduced in section II of chapter IV.

Continuity Equations and Their Solution

Let $p(n,r)$ denote the steady state probability that there are n units in the queue, the unit in the arrival channel being in the rth branch. Also let $q(0,r)$ denote the probability that there is no unit in the system, the unit in the arrival channel being in the rth branch. Assuming steady state conditions, these probabilities satisfy the following continuity equations:

$$-\left[\lambda_r(n) + \mu(n)\right] p(n,r) + \sigma_r \sum_{j=1}^{k} \lambda_j\, p(n-1)\, p(n-1,j) + \mu(n+S)\, p(n+S,\, r) = 0 \tag{39}$$

$$(n = 1, 2, \ldots, N-S)$$

$$- \left[\lambda_r(0) + \mu(0)\right] p(0,r) + \sigma_r \sum_{s=1}^{k} \lambda_s(0)\, q(0,s) + \sum_{\ell=1}^{S} p(\ell, r)\mu(\ell) = 0 \tag{40}$$

$$- \lambda_r(0)\, q(0,r) + \mu(0)\, p(0,r) = 0 \tag{41}$$

$$- \left[\lambda_r(m) + \mu(m)\right] p(m,r) + \sigma_r \sum_{j=1}^{k} \lambda_j(m-1)\, p(m-1,j) = 0 \tag{42}$$

$$(m = N-S+1, \ldots, N-1)$$

$$- \left[\lambda_r(N) + \mu(N)\right] p(N,r) + \sigma_r \sum_{j=1}^{k} \lambda_j(N-1)\, p(N-1,j) + \sigma_r \sum_{j=1}^{k} \lambda_j(N)\, p(N,j) = 0 \tag{43}$$

where the equations (39) through (43) are valud for $r = 1, 2, \ldots, k$.

From equation (41), we get

$$q(0,r) = \frac{\mu(0)\, p(0,r)}{\lambda_r(0)} \tag{44}$$

Substituting the value of $q(0,r)$ from (44) in (40), we get the matrix equation

$$\underset{\sim}{A}\underset{\sim}{P} = \underset{\sim}{Q} \tag{45}$$

where $\underset{\sim}{A}$ is the matrix given by

$$\underset{\sim}{A} = \| a_{ij} \|$$

such that

$$a_{ij} = - \mu(0)\sigma_i, \qquad (i \neq j)$$

$$a_{ii} = \lambda_i(0) + \mu(0)(1 - \sigma_i)$$

and $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are the column matrices given by

$$P = \left[p(0,1),\ p(0,2), \ldots\ldots\ldots,\ p(0,k) \right]$$

and

$$Q = \left[\sum_{\ell=1}^{S} \mu(\ell) p(\ell,1),\ \sum_{\ell=1}^{S} \mu(\ell)\, p(\ell,2), \ldots\ldots,\ \sum_{\ell=1}^{S} \mu(\ell) p(\ell,k) \right]$$

respectively.

We observe that $\underline{A}^{-1}$ is given by

$$\underline{A}^{-1} = \| a'_{ij} \|$$

such that

$$a'_{ij} = \frac{\mu(0) \sigma_i}{B\, A_i\, A_j}, \qquad (i \neq j)$$

$$a'_{ii} = \frac{B + \sigma_i / A_i}{B\, A_i}$$

where

$$A_i = \lambda_i(0) + \mu(0)$$

and

$$B = 1 - \mu(0) \sum_{r=1}^{k} (\sigma_r / A_r).$$

Pre-multiplying both sides of equation (45) by $\underline{A}^{-1}$, we obtain for

$i = 1, 2, \ldots\ldots, k$

$$p(0,i) = \frac{1}{A_i} \left[\sum_{\ell=1}^{S} \mu(\ell)\, p(\ell,i) + \frac{\mu(0) \sigma_i \sum_{\ell=1}^{S} \sum_{j=1}^{k} \mu(\ell) p(\ell,j) / A_j}{B} \right] \tag{46}$$

Now, summing up equation (39) over r and rearranging terms, we get

$$\mu(n) \sum_{r=1}^{k} p(n,r) - \sum_{r=1}^{k} \lambda_r(n-1)\, p(n-1,r) = \mu(n+1) \sum_{r=1}^{k} p(n+1,r) - \sum_{r=1}^{k} \lambda_r(n) p(n,r) \tag{47}$$

Summing up equation (47) over $n = 1,2,\ldots,n-1$, we get

$$\sum_{r=1}^{k} \lambda_r(n-1)\, p(n-1,r) = \sum_{i=n+1}^{n+S-1} \sum_{r=1}^{k} \mu(i)\, p(i,r) + \sum_{r=1}^{k} \overset{\mu}{\cancel{\lambda_r}}(n)\, p(n,r) \tag{48}$$

where we have used the fact that

$$\sum_{l=1}^{S} \sum_{r=1}^{k} \mu(l)\, p(l,r) = \sum_{r=1}^{k} \lambda_r(0)\, p(0,r) \tag{49}$$

in view of (40) and (41).

Using the value of $\sum_{r=1}^{k} \lambda_r(n-1)\, p(n-1,r)$ from (48) in (39), we get

$$\left[\lambda_r(n) + \mu(n)\right] p(n,r) - \mu(n)\, \sigma_r \sum_{r=1}^{k} p(n,r) = B_{r,n} \tag{50}$$

where

$$B_{r,n} = \sigma_r \sum_{i=n+1}^{n+S-1} \sum_{r=1}^{k} \mu(i)\, p(i,r) + \mu(n+S)\, p(n+S,r).$$

The matrix formed by the coefficients of $p(n,i)$ in (50) is of the same form as the matrix $\underline{A}$ in the calculation of $p(0,i)$, and therefore, as there, we have the solution

$$p(n,i) = \frac{1}{C_{i,n}} \left[B_{i,n} + \frac{1}{D_n} \left\{ \mu(n)\, \sigma_i \sum_{j=1}^{k} (B_{j,n}/C_{j,n}) \right\} \right] \tag{51}$$

$$(n = 1,2,\ldots,N-S;\ i = 1,2,\ldots,k)$$

where

$$C_{i,n} = \lambda_i(n) + \mu(n),$$

and

$$D_n = 1 - \mu(n) \sum_{j=1}^{k} (\sigma_j / C_{j,n}).$$

Now summing up equation (42) over r, we get

$$\sum_{j=1}^{k} \lambda_j(n-1)\, p(n-1,j) = \sum_{j=1}^{k} C_{j,n}\, p(n,j) \tag{52}$$

Substituting the value of $\sum_{j=1} \lambda_j(m-1)\ p(m-1,j)$ from (52) in (42),

we get

$$-C_{r,m}\ p(m,r) + \sigma_r \sum_{j=1}^{k} C_{j,m}\ p(m,j) = 0 \qquad (53)$$

$$(m = N-S+1, \ \ldots.N-1)$$

For m = N-S+1, equation (53) becomes

$$\underset{\sim}{C}\underset{\sim}{R} = -C_{k,N-S+1}\ p(N-S+1,k)\ \underset{\sim}{S} \qquad (54)$$

where $\underset{\sim}{C}$ is the $(k-1)\times(k-1)$ matrix

$$\underset{\sim}{C} = \left\| c_{ij} \right\|$$

such that

$$c_{ij} = \sigma_i\ C_{j,N-S+1}, \quad (i \neq j)$$

$$c_{ii} = (\sigma_i - 1)\ C_{i,N-S+1}$$

and $\underset{\sim}{R}$ and $\underset{\sim}{S}$ are the column matrices

$$R = \left[p(N-S+1,1), \ \ldots\ldots\ldots, \ p(N-S+1,k-1)\right]$$

and

$$S = \left[\sigma_1, \ \sigma_2, \ldots\ldots\ldots\ldots, \sigma_{k-1}\right]$$

respectively.

Now, $\underset{\sim}{C}^{-1}$ is given by

$$\underset{\sim}{C}^{-1} = \left\| c'_{ij} \right\|$$

such that

$$c'_{ij} = -\frac{\sigma_i}{\sigma_k\ C_{i,N-S+1}}, \quad (i \neq j)$$

$$c'_{ii} = -\frac{(\sigma_i + \sigma_k)}{\sigma_k\ C_{i,N-S+1}}$$

Pre-multiplying (54) by $\underset{\sim}{C}^{-1}$, we obtain

$$p(N-S+1,r) = \frac{\sigma_r \, C_{k,N-S+1} \, p(N-S+1,k)}{\sigma_k \, C_{r,N-S+1}} \tag{55}$$

$$(r = 1,2,\ldots,k-1)$$

Now, adding all the equations represented by (39), (40) and (42), we get

$$\sum_{r=1}^{k} \lambda_r(N-1) \, p(N-1,r) = \mu(N) \sum_{r=1}^{k} p(N,r) \tag{56}$$

Substituting back the value of $\sum_{r=1}^{k} \lambda_r(N-1) \, p(N-1,r)$ from (56) in (43), we get

$$\sigma_r \sum_{j=1}^{k} C_{j,N} \, p(N,j) - C_{r,N} \, p(N,r) = 0 \tag{57}$$

Equation (57) may be solved for $p(N,r)$ exactly as we solved $p(N-S+1,r)$ above, because the matrix formed by the coefficients is similar. The solution, therefore, is

$$p(N,r) = \frac{\sigma_r \, C_{k,N} \, p(N,k)}{\sigma_k \, C_{r,N}} \tag{58}$$

$$(r = 1,2,\ldots,k-1)$$

Also from equation (45), we have

$$p(N,k) = \frac{\sigma_k \sum_{j=1}^{k} \lambda_j(N-1) \, p(N-1,j)}{C_{k,N} - C_{k,N} \sum_{j=1}^{k} \sigma_j \lambda_j(N)/C_{j,N}} \tag{59}$$

Also from equation (42), we have

$$p(m,r) = \frac{\sigma_r \sum_{j=1}^{k} \lambda_j(m-1) \, p(m-1,j)}{C_{r,m}} \tag{60}$$

$$(m = N-S+1,\ldots,N-1)$$

Thus we have expressed all the probabilities, $p(n,r)$ and $q(0,r)$ for $n = 0,1,2,\ldots,N$; $r = 1,2,\ldots,k$ in terms of one unknown probability, namely, $p(N-S+1,k)$. This unknown probability may now be evaluated by using the normalising condition, i.e.

$$\sum_{n=0}^{N} \sum_{r=1}^{k} p(n,r) + \sum_{r=1}^{k} q(0,r) = 1 \tag{61}$$

and hence all the probabilities are completely known.

The recurrence relations contained in equations (44), (46), (51), (55), (58), (59), (60) and (61) are the required recurrence relations which help us evaluate the probabilities uniquely.

<u>Bulk Service Queuing System M/M/1 with State Dependant Parameters</u>

If in the recurrence relations obtained above, we substitute

$$\sigma_r = \delta_{rk}, \quad \sum_{i=1}^{k} p(n,i) = p(n),$$

$$q(0) = \sum_{r=1}^{k} q(0,r)$$

we obtain the recurrence relations for the queuing system M/M/1 with bulk service and state dependant parameters. These relations are:

$$q(0) = \frac{\mu(0)\ p(0)}{\lambda(0)} \tag{62}$$

$$p(0) = \frac{\sum_{s=1}^{k} \mu(s)\, p(s)}{\lambda(0)} \tag{63}$$

$$p(n) = \frac{\sum_{i=n+1}^{n+k} \mu(i)\, p(i)}{\lambda(n)} \tag{64}$$

$$(n = 1,2,\ldots,N-S)$$

$$p(m) = \frac{\lambda(m-1)\; p(m-1)}{\lambda(m) + \mu(m)}$$

$$= p(N-S) \prod_{i=N-S}^{m-1} \alpha_i \qquad (65)$$

$$(m = N-S+1, \ldots, N-1)$$

$$\text{where} \quad \alpha_{i-1} = \frac{\lambda(i-1)}{\lambda(i) + \mu(i)}$$

$$p(N-S) = \frac{\mu(N)\; p(N)}{\lambda(N-S) - \sum_{m=N-S+1}^{N-1} \left(\mu(m) \prod_{i=N-S}^{m-1} \alpha_i \right)} \qquad (66)$$

$$\sum_{i=0}^{N} p(i) + q(0) = 1. \qquad (67)$$

Numerical Calculations

The recurrence relations for both queuing systems were tried on IBM 1620 digital computer and in a number of cases the probabilities were evaluated. Here we present the graphs for

(i) probability of no delay, i.e. $\sum_{r=1}^{k} q(0,r)$

(ii) mean number of units in the queue

by assuming that

$$\lambda_r(n) = k\,\sigma_r\,\lambda(1 - n/N), \quad \text{balking}$$

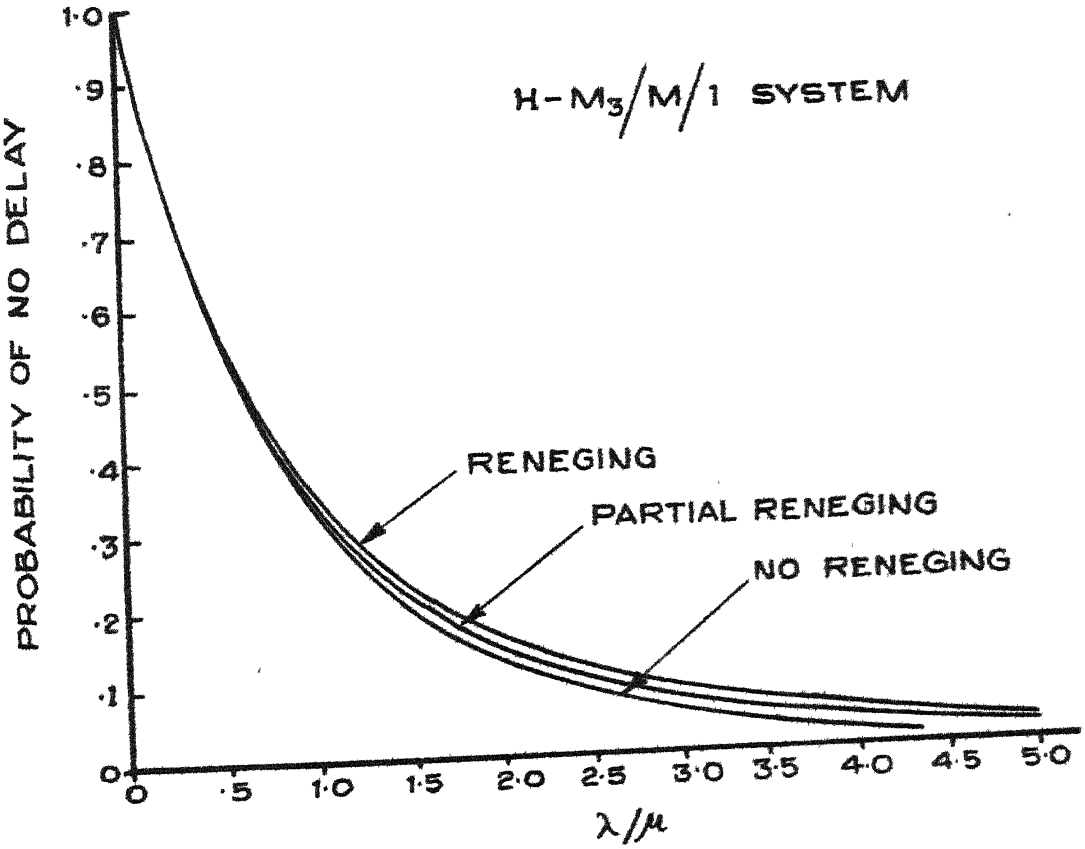
$$\mu(n) = \mu + (n-1)\alpha, \quad \text{reneging}$$

$$\text{or} \quad \left.\begin{aligned} \mu(n) &= \mu && \text{for } n = 1,2,3,4,5 \\ &= \mu + (n-1)\alpha && \text{for } n > 5 \end{aligned}\right\} \text{partial reneging,}$$

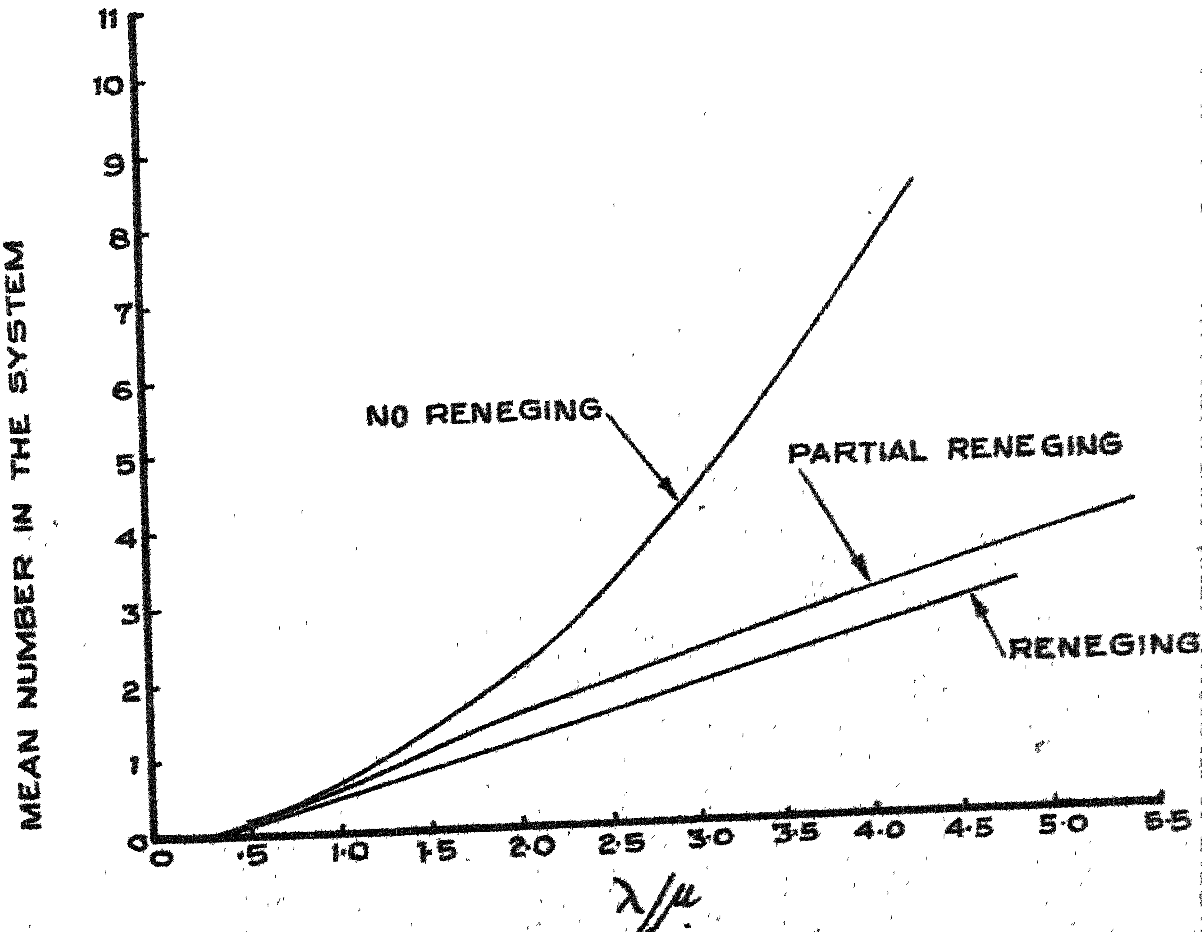
and $\mu = 6$, $\alpha = 1.2$, $N = 20$, $k = 3$, $S = 5$, $\sigma_1 = .22$, $\sigma_2 = .33$, $\sigma_3 = .45$.

From this numerical work, it is observed that for small values of λ the probability of no delay, p_0, say, is always high, whether the queue parameters depend upon the state of the system or not. For large values of λ, p_0 is too small in all cases. A difference is noted in the value of p_N, i.e. probability that the system is full. When the parameters are not constant, it is negligible whether λ is small or large. But p_N is appreciable when the parameters are constant. Thus with variable parameters, the probabilities fall of very rapidly and there is no necessity of fixing an upper bound on queue length.

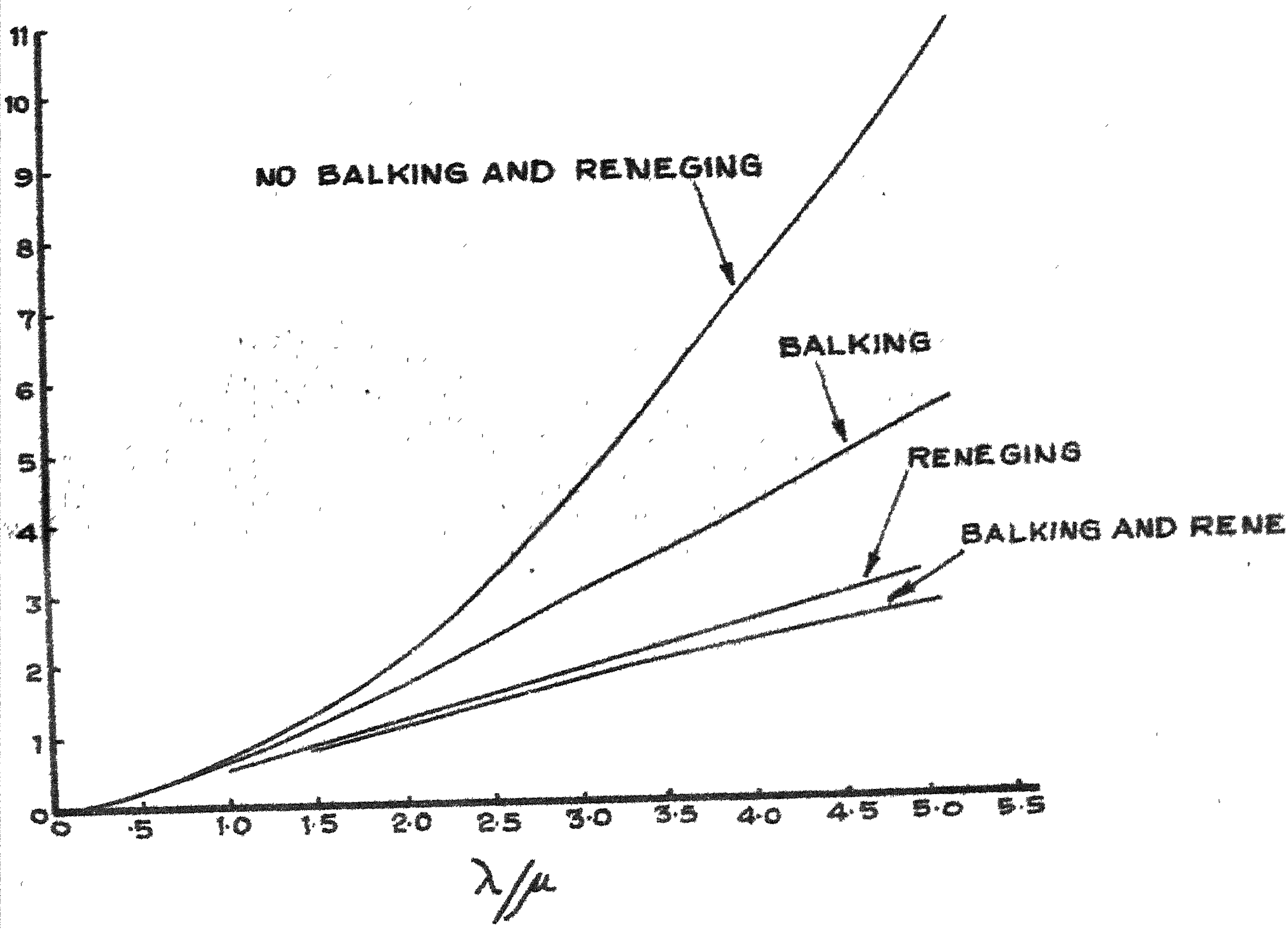
One other important difference to be noted is that while studying bulk service, we cannot deduce the results for multiple server bulk service system even after we have obtained results with arbitrary state functions whereas in the ordinary case when units are served singly, this is obviously possible, as for example we deduced the results due to Nishida (1962) in section II of chapter IV. However, we fail to deduce directly the results due due to Arora (1964a) from the results arrived at in the sub-section — Queuing System M/M/1 — of this section.


H-M3/M/1 SYSTEM
PROBABILITY OF NO DELAY
RENEGING
PARTIAL RENEGING
NO RENEGING
λ/μ
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
0
·5
1·0
1·5
2·0
2·5
3·0
3·5
4·0
4·5
5·0

H-M$_3$/M/1 SYSTEM



M/M/1 SYSTEM


11
10
9
8
7
6
5
4
3
2
1
0
NO BALKING AND RENEGING
BALKING
RENEGING
BALKING AND RENE
0
·5
1·0
1·5
2·0
2·5
3·0
3·5
4·0
4·5
5·0
5·5
λ/μ

CHAPTER VI

GRADED MULTIPLE QUEUES

So far we considered queuing systems in which only one server is available at the service facility except at one place in chapter IV. However, many workers have considered queuing systems in which more than one server is available at the service facility. Reference to this effect may be made to: Erlang (see Brockmeyer et. al. 1948), Molina (1927), Kolmogorov (1931), Kendall (1953), Fagen and Riordan (1955), Gumbel (1960), Homma (1956), Joffe and Ney (1960), Karlin and McGregor (1958), Keifer and Wolfowitz (1955), Saaty (1960), Ancker and Gafarian (1962), Haight (1958), Krishnamoorti (1963), Yuh Ming-I (1959), Hong (1962), Nishida (1962), Palm (1936, 1943), Wilkinson (1931), Disney (1962, 1963) etc.

The queuing systems considered by these workers are different either in that the arrival and/or service time distribution is different or rules for selecting the server are different etc.

One type of multiple server queuing system called arrangements in echelon by Palm (1936), and graded multiple systems by Wilkinson (1931) is characterised as follows: the servers are arranged in ordered groups, and a demand which can be served is served by a server in the group of lowest order having a server free. Thus the traffic input to a given service group is the overflow from its immediate predecessor (or predecessors).

Palm (1936) and Wilkinson (1931), however, considered such a system with losses only, i.e. when no queue is allowed in front of the various servers

An important fact to note about this problem, as mentioned in Khinchin (1960), is that the results do not depend upon whether the number of groups of servers available to the units is finite or infinite. This difference is important if the servers are not ordered.

Disney (1962) attempted to solve this problem when delays are allowed, i.e., when queues are allowed in front of the servers. To be more precise, he posed the following queuing problem:

Units arrive at a service station according to an orderly, stationary Poisson stream without after effects with parameter λ and are served at a number of service channels (numbered in order $1,2,\ldots,n$), each channel containing one server. The service time distribution for each of the channels is negative exponential with parameter μ. The waiting space in front of each of the service channels is fixed, say for $N_1, N_2, \ldots, N_n$ units in channels $1,2,\ldots,n$ respectively. The incoming unit first goes to channel 1. If it is packed to capacity it goes to channel 2, otherwise it takes its position in channel 1. Each incoming unit tests each of the n channels in the prescribed order until it finds some channel which can accomodate it. In case the incoming unit finds all the channels packed to capacity, it leaves without service and is thus 'lost' to the system.

Disney (1962) wrote the queue equations in the steady state case for the two channel case and solved the equations explicitly in two particular cases: (i) maximum units allowed in channel 1 is 1 and in channel 2 is 1; (ii) maximum units allowed in channel 1 is 1 and in channel 2 is 3. Disney (1963) again considered this problem when the maximum units allowed in channel 1 is 1 and in channel 2 is N and pointed out that this queuing model has applications in conveyor theory.

The author (1965), however, could solve this two channel queuing system when the maximum units allowed in channels 1 and 2 is M and N respectively by using the technique of generating functions. The queue size distribution is obtained and in the particular case when N = 1, some numerical work has also been done. The analysis of this system is reproduced below.

<u>Continuity Equations</u>

Let $p(k,j)$ denote the probability that channel 1 has a queue of length k units and channel 2 has a queue of length j units with $0 \leqslant k \leqslant M$ and $0 \leqslant j \leqslant N$, j and k taking integral values only. As given by Disney (1962, 1963), the continuity equations are:

$$-\lambda p(0,0) + \mu p(1,0) + \mu p(0,1) = 0 \tag{1}$$

$$-(\lambda + \mu)p(0,N) + \mu p(1,N) = 0 \tag{2}$$

$$-(\lambda + \mu)\, p(M,0) + \mu p(M,1) + \lambda p(M-1,0) = 0 \tag{3}$$

$$-2\mu\, p(M,N) + \lambda p(M-1,N) + \lambda p(M,N-1) = 0 \tag{4}$$

$$-(\lambda + \mu)\, p(0,j) + \mu p(1,j) + \mu p(0,j+1) = 0 \tag{5}$$

$$(j = 1,2,\ldots,N-1)$$

$$-(\lambda + \mu)p(k,0) + \mu p(k,1) + \lambda p(k-1,0) + \mu p(k+1,0) = 0 \tag{6}$$

$$(k = 1,2,\ldots M-1)$$

$$-(\lambda + 2\mu)p(M,j) + \lambda p(M-1,j) + \lambda p(M,j-1) + \mu p(M,j+1) = 0 \tag{7}$$

$$(j = 1,2,\ldots,N-1)$$

$$-(\lambda + 2\mu)\, p(k,N) + \lambda p(k-1,N) + \mu p(k+1,N) = 0 \tag{8}$$

$$(k = 1,2,\ldots,M-1)$$

$$-(\lambda + 2\mu)p(k,j) + \lambda p(k-1,j) + \mu p(k+1,j) + \mu p(k,j+1) = 0 \tag{9}$$

$$(k = 1,2,\ldots,M-1;\ j = 1,2,\ldots,N-1)$$

Solution of the Continuity Equations

Let us define the generating functions

$$F_j(x) = \sum_{k=1}^{M} p(k,j)\, x^k \tag{10}$$

Equations (1), (3) and (6) on using eqution (10) give

$$\left[-(\lambda+\mu) + \lambda x + \mu/x\right] F_0(x) + \mu F_1(x) + \lambda(x-1)\, p(0,0) + \mu\, p(0,1) - \lambda x^{M+1}\, p(M,0) = 0 \tag{11}$$

Equations (2), (4) and (8) on using equations (10) give

$$\left[-(\lambda+2\mu) + \lambda x + \mu/x\right] F_N(x) + \lambda x^M (1-x)\, p(M,N) + \lambda x\, p(0,N) - (\lambda+\mu)\, p(0,N) + \lambda x^M\, p(M,N-1) = 0 \tag{12}$$

Equations (5), (7) and (9) on using equation (10) give

$$\left[-(\lambda+2\mu) + \lambda x + \mu/x\right] F_j(x) + \mu F_{j+1}(x) + \lambda x\, p(0,j) - \lambda x^{M+1}\, p(M,j) + \lambda x^M\, p(M,j-1) - (\lambda+\mu)\, p(0,j) + \mu\, p(0,j+1) = 0 \tag{13}$$

$$(j = 1,2,\ldots,N-1).$$

Again define the generating function

$$G(x,y) = \sum_{j=1}^{N} F_j(x)\, y^j \tag{14}$$

Equations (12), (13) on using equations (11) and (14) give

$$\left[-(\lambda+2\mu)+\lambda x+\mu/x+\mu/y\right] G(x,y) + \lambda(x-1)\, p(0,0) +$$

$$\lambda x^{M}(y-x)\, p(M,0) + \left[-(\lambda+\mu)+\lambda x+\mu/x\right] F_0(x) +$$

$$\left[\lambda(x-1)+\mu(1/y-1)\right] \sum_{j=1}^{N} p(0,j)\, y^{j} + \lambda x^{M} y^{N}(1-y)\, p(M,N)$$

$$+ \lambda x^{M}(y-x) \sum_{j=1}^{N} p(M,j)\, y^{j} = 0 \qquad (15)$$

Let us now substitute

$$y = \frac{\mu x}{(\lambda+2\mu)x-\mu-\lambda x^{2}} = z \text{ (say)}$$

in equation (15), then

$$F_0(x) = \sum_{k=1}^{M} p(k,0)\, x^{k} = \frac{x}{\mu-\lambda x}\Big[\lambda\, p(0,0) + \frac{\mu}{x}\sum_{j=1}^{N} p(0,j)\, z^{j} +$$

$$\frac{1}{(\lambda+2\mu)x-\mu-\lambda x^{2}}\Big\{\lambda x^{M+1}(\lambda x-2\mu)\, p(M,0)$$

$$+ \lambda x^{M} z^{N}(\mu-\lambda x)\, p(M,N) +$$

$$\lambda x^{M+1}(\lambda x-2\mu)\sum_{j=1}^{N} p(M,j)\, z^{j}\Big\}\Big]$$

$$(16)$$

The left hand side of equation (15) is a polynomial and is hence analytic. So must, therefore, be the right hand side also. The denominator on the right hand side is a polynomial of degree 2N + 1. In order that the right hand side be analytic, the numerator must cancel the 2N + 1 zeros of the denominator. This condition gives rise to 2N + 1 homogeneous linear equations involving the 2N + 2 unknown probabilities involved in the numerator of equation (16), viz. p(0,0), p(M,0), p(M,j), p(0,j) for j = 1, 2,.....,N. Also the

normalizing condition is

$$\sum_{k=0}^{M} \sum_{j=0}^{N} p(k,j) = 1$$

or

$$G(1,1) + \sum_{j=0}^{N} p(0,j) + \sum_{k=1}^{M} p(k,0) = 1$$

or

$$\mu \sum_{j=0}^{N} p(0,j) - \lambda \sum_{j=0}^{N} p(M,j) = \mu - \lambda. \tag{17}$$

Thus the 2N + 1 equations as conditions of analyticity and equation (17) are sufficient to solve for the 2N + 2 unknown probabilities. After these probabilities are determined, $F_0(x)$ is immediately known and hence $G(x,y)$ also from equation (15). This in principle atleast determines all the probabilities.

Mean Number of Units

$$\text{Mean number of units} = \sum_{k=0}^{M} \sum_{j=0}^{N} (k+j)\, p(k,j)$$

$$= \left.\frac{\partial G(x,1)}{\partial x}\right|_{x=1} + \left.\frac{\partial G(1,y)}{\partial y}\right|_{y=1} + \sum_{j=1}^{N} j\, p(0,j) + \sum_{k=1}^{M} k\, p(k,0)$$

$$= \frac{\lambda(M+\lambda/\mu)}{\lambda-\mu} \sum_{j=0}^{N} p(M,j) + \frac{\lambda}{\mu} \sum_{j=0}^{N-1} (j+1)\, p(M,j) + \frac{\lambda^2}{\mu(\mu-\lambda)} \tag{18}$$

Particular Case

If N = 1, then equation (16) gives

$$F_0(x) = \frac{x}{(\mu - \lambda x)\left[(\lambda + 2\mu)x - \mu - \lambda x^2\right]} \Big[\lambda\left\{(\lambda + 2\mu)x - \mu - \lambda x^2\right\} \cdot$$

$$p(0,0) + \mu^2 p(0,1) + \lambda x^{M+1}(\lambda x - 2\mu)\, p(M,0) -$$

$$\lambda x^{M+1} \mu\, p(M,1) \Big] \tag{19}$$

The four equations for the solution of $p(0,0)$, $p(0,1)$, $p(M,0)$ and $p(M,1)$ in this case are

$$\mu^2 p(0,0) + \mu^2 p(0,1) + \lambda A_1\, p(M,0) - \lambda \mu B_1\, p(M,1) = 0 \tag{20}$$

$$\mu^2 p(0,1) + \lambda A_i\, p(M,0) - \lambda \mu B_i\, p(M,1) = 0 \tag{21}$$

$$(i = 2,3)$$

$$\mu\, p(0,0) + \mu\, p(0,1) - \lambda\, p(M,0) - \lambda\, p(M,1) = \mu - \lambda \tag{22}$$

where

$$x_1 = \mu / \lambda$$

$$x_{2,3} = \frac{1}{2\lambda}\left[\lambda + 2\mu \pm \sqrt{\lambda^2 + 4\mu^2}\right]$$

$$A_i = x_i^{M+1}(\lambda x_i - 2\mu), \quad i = 1,2,3,$$

$$B_i = x_i^{M+1}, \qquad i = 1,2,3.$$

Solving the equations (20), (21), (22) for the four unknowns, one gets

$$p(0,0) = \frac{\mu - \lambda}{\mu D}\left[A_1 (B_3 - B_2) + A_2(B_1 - B_3) + A_3(B_2 - B_1)\right] \qquad (23)$$

$$p(0,1) = \frac{\mu - \lambda}{\mu D}(A_2 B_3 - A_3 B_2) \qquad (24)$$

$$p(N,0) = \frac{\mu(\mu - \lambda)}{\lambda D}(B_2 - B_3) \qquad (25)$$

$$p(N,1) = \frac{\mu - \lambda}{\lambda D}(A_2 - A_3) \qquad (26)$$

where

$$D = (A_3 - A_2)(1 - B_1) + (\mu + A_1)(B_3 - B_2).$$

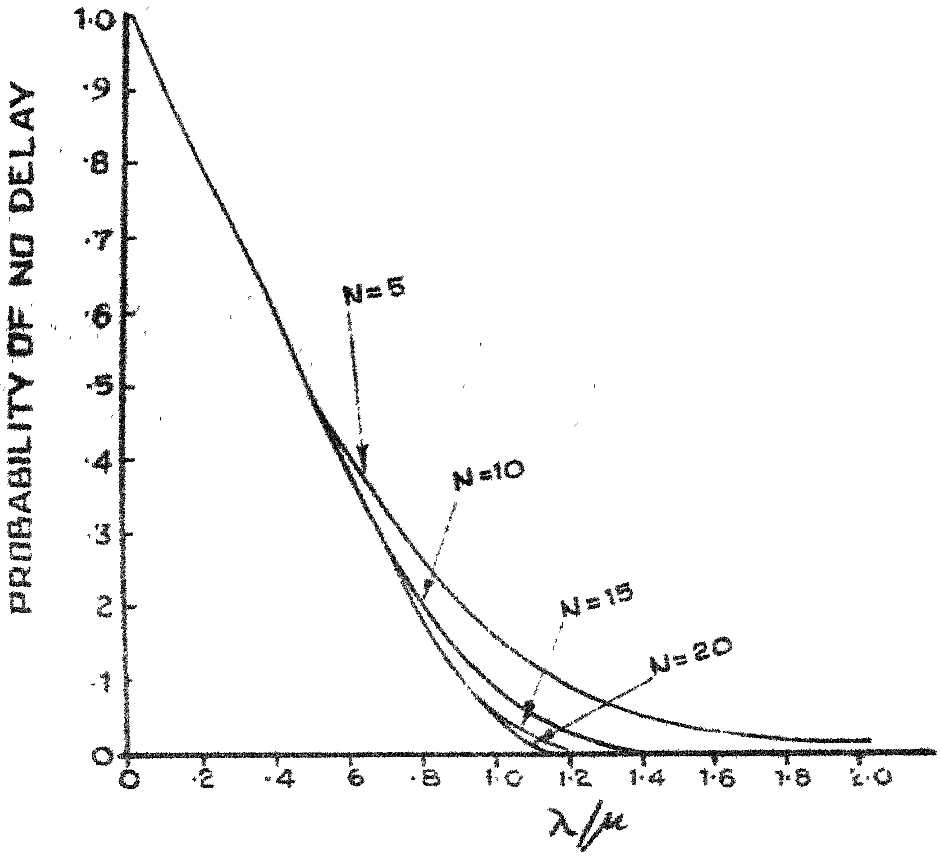
Numerical Work

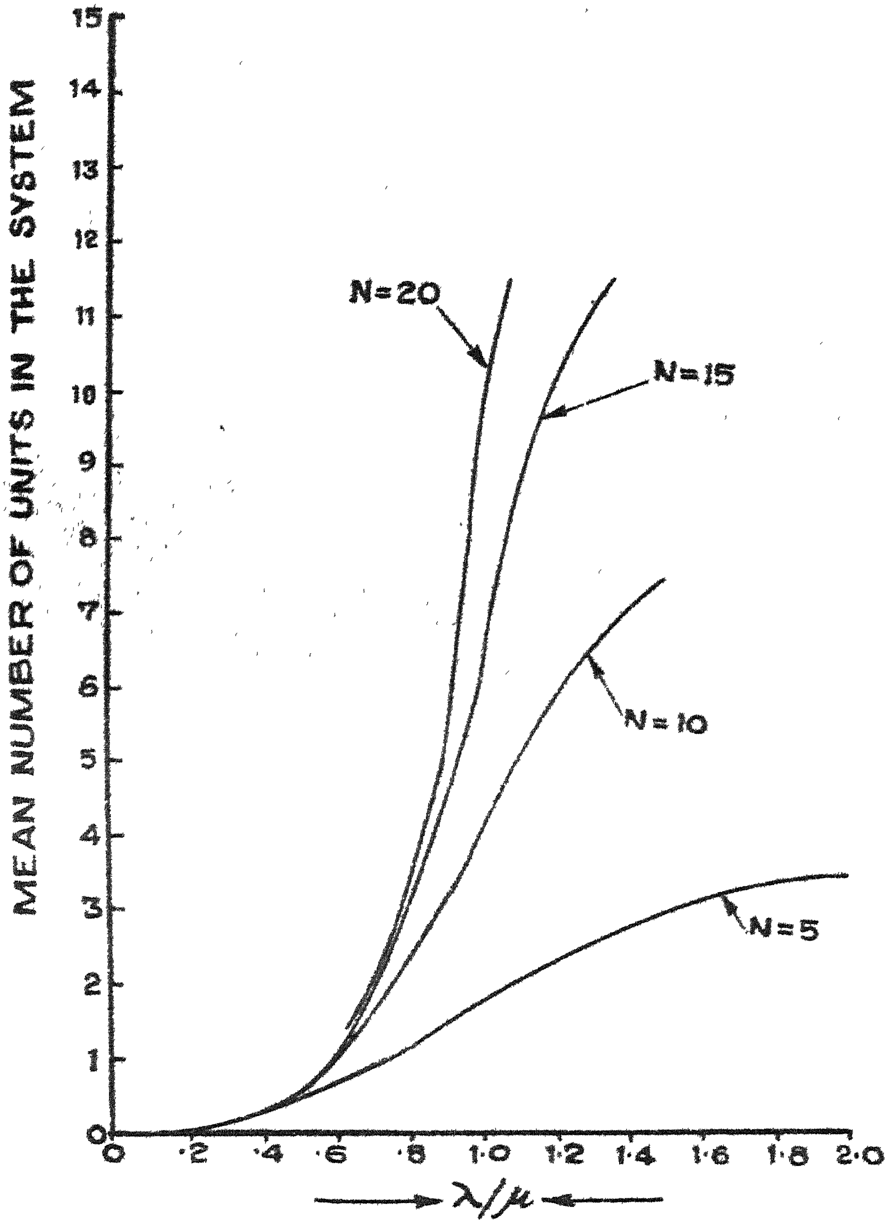
When $\mu = 5.0$ and $N = 5$, 10, 15, and 20, the graphs showing the behaviour of

(i) probability of no delay, i.e. $p(0,0)$

(ii) mean number of units in the queue, from equation (18),

against $\rho = \lambda / \mu$ have been drawn.


PROBABILITY OF NO DELAY
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
N=5
N=10
N=15
N=20
0
·2
·4
·6
·8
1·0
1·2
1·4
1·6
1·8
2·0
λ/μ


MEAN NUMBER OF UNITS IN THE SYSTEM
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
N=20
N=15
N=10
N=5
0
·2
·4
·6
·8
1·0
1·2
1·4
1·6
1·8
2·0
λ/μ

APPENDIX

In this appendix we give the numerical calculations from which the graphs appearing in the main thesis were drawn. These numerical calculations were done on the IBM 1620 digital computer available at the Indian Institute of Technology, Kanpur. The queue parameters for which these calculations were done have already appeared at the relevant places in the thesis. Here we simply refer to the pages on which the corresponding graphs appear.

TABLE I

Probability of no Delay (page 39)

	$\lambda_2 = 2$	$\lambda_2 = 4$	$\lambda_2 = 6$	$\lambda_2 = 8$	$\lambda_2 = 10$	$\lambda_2 = 12$
			Probability of No Delay			
.2	.96665618	.93333131	.89999904	.86666617	.83333320	.79999987
.4	.94284729	.88571355	.82857114	.77142845	.71428568	.65714275
.6	.92499805	.84999973	.77499966	.60000085	.62499996	.54999992
.8	.91110786	.82222135	.73333300	.64444131	.55555550	.46666692
1.0	.89999661	.79999960	.63999981		.49999998	.46666659
1.2	.89090706	.78181800	.67272732	.56363622	.45454538	.40000011
1.4	.88333195	.76666656	.64999974	.53333326	.41666672	.34545673
1.6	.87692260	.75384585	.63076907	.50769220	.38461579	.30001429
1.8	.87142725	.74285695	.61428559	.48571429	.35714461	.26159920
2.0	.86666582	.73333323	.60000002	.46666670	.33339845	.22876327
2.2	.86249865	.72499976	.58749995	.44999996	.31251248	.20048497
2.4	.85882337	.71764693	.57647062	.43559418	.29414402	.17603262
2.6	.85555559	.71111103	.56666664	.42222239	.27782778	.15486406
2.8	.85263066	.70526299	.55789466	.41052657	.25324480	.13635791
3.0	.84999942	.70000006	.54999995	.40000055	.25014026	.12074519
3.2	.84761892	.69523890	.54285709	.39047708	.23850800	.10712390
3.4	.84545454	.69090912	.53636358	.38181956	.22758108	.09540541
3.6	.84347812	.68695646	.53043472	.37391514	.21781677	.08533145
3.8	.84166644	.68333329	.52499991	.36666967	.20890512	.07866965
4.0	.83999945	.67999999	.51999998	.36000421	.20074175	.06921384

TABLE 2

Mean Number of Units in the System (page 40)

	$\lambda_2=2$	$\lambda_2=4$	$\lambda_2=6$	$\lambda_2=8$	$\lambda_2=10$	$\lambda_2=12$
	Mean Number of Units in the System					
.2	.0334	.0681	.1044	.1430	.1841	.2284
.4	.0578	.1193	.1867	.2622	.3485	.4485
.6	.0763	.1597	.2548	.2672	.5040	.6785
.8	.0910	.1928	.3134	.4628	.6564	.9203
1.0	.1029	.2206	.3649		.8090	1.1900
1.2	.1129	.2447	.4112	.6360	.9633	1.4917
1.4	.1213	.2657	.4532	.7162	1.1203	1.8316
1.6	.1286	.2843	.4918	.7929	1.2803	2.2114
1.8	.1349	.3010	.5274	.8666	1.4433	2.6442
2.0	.1405	.3161	.5605	.9375	1.6091	3.1191
2.2	.1455	.3299	.5914	1.0058	1.7774	3.6339
2.4	.1500	.3425	.6204	1.0716	1.9478	4.1791
2.6	.1541	.3542	.6475	1.1351	2.1196	4.7428
2.8	.1577	.3649	.6731	1.1962	2.2924	5.3119
3.0	.1611	.3750	.6972	1.2552	2.4654	5.8747
3.2	.1642	.3843	.7200	1.3121	2.6379	6.4214
3.4	.1671	.3930	.7416	1.3670	2.8094	6.9448
3.6	.1697	.4012	.7621	1.4199	2.9792	7.4400
3.8	.1722	.4089	.7815	1.4709	3.1468	7.9044
4.0	.1745	.4162	.7999	1.5201	3.3116	8.3370

TABLE 3

Mean Waiting Time (page 41)

	$\lambda_2=2$	$\lambda_2=4$	$\lambda_2=6$	$\lambda_2=8$	$\lambda_2=10$	$\lambda_2=12$
	Mean Waiting Time					
.2	.00065298	.00224395	.00448580	.00725475	.01044987	.01420378
.4	.00127870	.00440217	.00892144	.01475121	.02198040	.03082921
.6	.00187852	.00647244	.01327769	.02241851	.03440410	.05017780
.8	.00245592	.00845413	.01753061	.03018637	.04770328	.07256767
1.0	.00300562	.01034778	.02166123		.06180339	.09834825
1.2	.00353496	.01215478	.02565589	.04576820	.07661891	.12791097
1.4	.00404289	.01387735	.02950525	.05347418	.09206470	.16166608
1.6	.00453050	.01551831	.03320404	.06106645	.10605825	.19994880
1.8	.00499867	.01708070	.03675009	.06851329	.12452060	.24286145
2.0	.00544841	.01856796	.04014378	.07579077	.14137493	.29012489
2.2	.00588051	.01998357	.04338748	.08288172	.15854447	.34102686
2.4	.00629605	.02133105	.04648502	.08977461	.17955096	.39449957
2.6	.00669564	.02261403	.04944150	.09646244	.19351399	.44928892
2.8	.00706021	.02383586	.05226140	.10294177	.21115216	.50414316
3.0	.00745035	.02400002	.05495169	.10921205	.22878455	.55796070
3.2	.00780683	.02610966	.05751809	.11527480	.24633324	.60987456
3.4	.00815037	.02718793	.05996678	.12113315	.26372531	.65922735
3.6	.00848148	.02817774	.06230384	.12679140	.28089466	.70578162
3.8	.00880086	.02914193	.06453520	.13225465	.29778316	.74921879
4.0	.00910895	.03006306	.06666664	.13752864	.31434146	.78955232

TABLE 4

Probability of No Delay (page 49)

	$\lambda_2=6$	$\lambda_2=8$	$\lambda_2=10$	$\lambda_2=12$
	Probability of no delay			
.2	.00000000	.00000000	.00000000	.00000000
.4	.00000000	.00000000	.00000024	.00002165
.6	.00000000	.00000092	.00013450	.00699909
.8	.00000001	.00003283	.00525[illegible]7	.08194156
1.0	.00000040	.00053549		.20177497
1.2	.00000402	.00361931	.10144[illegible]27	.30939020
1.4	.00002318	.01319002	.16986[illegible]3[illegible]	.40[illegible]07476
1.6	.00009107	.03125256	.2319353[illegible]	.47694847
1.8	.00027030	.05570869	.2862223[illegible]	.54296800
2.0	.00064930	.02292077	.3335899[illegible]	.60000549
2.2	.00132552	.11018822	.3751458[illegible]	.65000323
2.4	.00238336	.13612565	.4118556[illegible]	.69411971
2.6	.00387665	.16019763	.4445056[illegible]	.73333467
2.8	.00582143	.18229919	.4757277[illegible]	.76842208
3.0	.00819871	.20231398	.50003256	.80000078
3.2	.01096320	.22099627	.5238348[illegible]	.82857210
3.4	.01405391	.23791739	.5454748[illegible]	.85454594
3.6	.01740376	.25344322	.5652234[illegible]	.87822613
3.8	.02094650	.26772559	.5833475[illegible]	.90000026
4.0	.02462131	.28089348	.60001206	.92000023

TABLE 5

Mean Number of Units in the System (page 50)

	$\mu_2=4$	$\mu_2=6$	$\mu_2=8$	$\mu_2=10$	$\mu_2=12$
	Mean Number of Units in the System				
.2	19.85	19.77	19.67	19.55	19.40
.4	19.73	19.55	19.29	18.89	18.21
.6	19.63	19.32	18.79	17.65	14.41
.8	19.53	19.07	18.07	14.82	7.78
1.0	19.44	18.80	16.97		4.57
1.2	19.36	18.50	15.32	6.99	3.41
1.4	19.28	18.16	13.22	5.16	2.88
1.6	19.20	17.78	11.07	4.20	2.59
1.8	19.13	17.36	9.25	3.66	2.41
2.0	19.06	16.89	7.86	3.31	2.29
2.2	18.99	16.39	6.84	3.08	2.20
2.4	18.92	15.86	6.09	2.92	2.14
2.6	18.86	15.33	5.64	2.79	2.09
2.8	18.80	14.79	5.13	2.69	2.04
3.0	18.74	14.26	4.81	2.62	2.01
3.2	16.68	13.75	4.56	2.56	1.98
3.4	16.65	13.27	4.35	2.51	1.96
3.6	18.58	12.82	4.19	2.46	1.94
3.8	16.52	12.39	4.05	2.43	1.92
4.0	18.47	12.00	3.93	2.39	1.91

TABLE 6

Probability of Loss (page 51)

	$\lambda_2=2$	$\lambda_2=4$	$\lambda_2=6$	$\lambda_2=8$	$\lambda_2=10$	$\lambda_2=12$
	Probability of Loss					
.2	.93333319	.86666662	.79999969	.73333321	.66666660	.59999983
.4	.88571420	.71428380	.65714274	.54285703	.42857155	.31430721
.6	.84999986	.69999984	.54999991	.40000046	.25013439	.10699904
.8	.82222212	.64444452	.46666652	.28892165	.2501347[illegible]	.01527487
1.0	.79999995	.59999997	.40000031	.20053541	.	.00177492
1.2	.78181812	.56363624	.34545851	.13089200	.01053512	.00029923
1.4	.76666668	.53333320	.30002312	.07985661	.00319661	.00007473
1.6	[illegible]	.50769220	.26163953	.04663710	.001[illegible]6605	.00002540
1.8	[illegible]	.48571426	.22684168	.02713719	.00050802	.00001087
2.0	[illegible]	.46666659	.20064922	.016[illegible]339	.00025657	.00000552
2.2	.72499987	.45000001	.17632500	.01018917	.000[illegible]4589	.00000319
2.4	.71764699	.43529415	.15532447	.00[illegible]1387	.00009095	.00000203
2.6	.71111092	.42222247	.13720[illegible]	.00[illegible]4204	.00005109	.00000139
2.8	.70526301	.41052636	.12161089	.00335181	.00004352	.00000202
3.0	.70000001	.40000091	.10819867	.00251398	.00003254	.00000077
3.2	.69523798	.39047781	.09667744	.00194861	.00002538	.00000061
3.4	.69090899	.38182065	.08676115	.00155371	.00002028	.00000050
3.6	.68695636	.37391677	.07827325	.00126931	.00001670	.00000042
3.8	.68333320	.36667190	.07094641	.00105809	.00001406	.00000036
4.0	.67999982	.36000713	.06462127	.00089945	.00001207	.00000031

TABLE 7

Probability of No Delay (page 106)

	b=-1, c=-1	b=-1, c=0	b=1, c=1	balking	b=0, c=0.
			Probability of No Delay		
.1666	.71428	.85082	.90909	.83766	.83333
.3333	.50000	.73018	.80000	.68515	.66666
.5000	.33337	.63089	.66671	.54467	.50005
.6666	.20102	.54802	.50170	.41887	.33480
.8333	.09993	.47807	.30815	.31042	.18193
1.0000	.03749	.41850	.13599	.22125	.07262
1.1666	.01104	.36757	.04365	.15176	.02208
1.3333		.32321	.01206	.10054	.00597
1.5000		.28487		.06470	
1.6666		.25145		.04073	
1.8333		.22220		.02527	
2.0000		.19652		.01554	
2.1666		.17393		.00953	
2.3333		.15401		.00586	
2.5000		.13642			
2.6666		.12085			
2.8333		.10708			
3.0000		.09486			

TABLE 8

Mean Number of Units in the System (page 107)

	b=-1, c=-1	b=-1, c=0	b=1, c=1	balking	b=0 c=0
	Mean Number of Units in the System				
.1666	4.57142	2.38674	1145454	2259734	2.66666
.3333	7.99999	4.31700	3.19999	5.03745	5.33332
.5000	10.66593	5.90575	5.33251	7.28522	7.99913
.6666	12.78360	7.23166	7.97274	9.29807	10.64314
.8333	14.40102	8.35077	11.06953	11.03319	13.08906
1.0000	15.40008	9.30398	13.82401	12.45989	14.83792
1.1666	15.82329	10.12205	15.30159	13.57167	15.64661
1.3333		10.82856	15.80703	14.39126	15.90447
1.5000		1144192		15.34816	
1.6666		11.97671		15.59561	
1.8333		12.44471		15.75119	
2.0000		12.85550		15.84738	
2.1666		13.21697		15.90627	
2.3333		13.53570			
2.5000		13.81722			
2.6666		14.06622			
2.8333		14.28669			
3.0000		14.48205			

TABLE 9

Mean Waiting Time (page 108)

	b = -1, c = -1	b = -1, c = 0	b = 1, c = 1	balking	b = 0, c = 0
			Mean Waiting Time		
.1666	.34162	.07955	.02252	.08657	.08888
.3333	.69458	.14390	.04517	.16791	.17777
.5000	1.13995	.19685	.06644	.24284	.26663
.6666	1.84023	.24105	.08263	.30993	.35477
.8333	3.03764	.27855	.08651	.36777	.43630
1.0000	4.64209	.31013	.07426	.41533	.49459
1.1666	5.02892	.33740	.05755	.45238	.52155
1.3333		.36095	.04682	.47970	.53014
1.5000		.38139		.49882	
1.6666		.39922		.51160	
1.8333		.41482		.51985	
2.0000		.42851		.52504	
2.1666		.44056		.52824	
2.3333		.45119		.53020	
2.5000		.46057			
2.6666		.46887			
2.8333		.47622			
3.0000		.48273			

TABLE 10

Fraction of Customers Lost (page 109)

Fraction of customers lost for b = 0, c = 0

	N = 5	N = 15	N = 10	N = 20
.1666	.00030	.00000	.00000	.00000
.3333	.01067	.00000	.00015	.00000
.5000	.07222	.00032	.11470	.00002
.6666	.24825	.00881	.04470	.00176
.8333	.58620	.09159	.21318	.04175
1.0000	1.09411	.43577	.62281	.33514
1.1666	1.74810	1.13251	1.28800	1.06522
1.3333	2.51234	2.03582	2.12733	
1.5000	3.35436		3.05635	
1.6666	4.24823			
1.8333	5.17653			
2.0000	6.12752			
2.1666	7.09355			
2.3333	8.06966			
2.5000	9.05260			

TABLE 11

Probability of No Delay (page 120)

	Three Server	Five Server	Reneging	b=-1, c=-1	b=0, c=0	b=-1, c=0
			Probability of No Delay			
.1666	.848239	.848252	.839064	.831058	.833333	.855855
.3333	.721948	.722108	.690934	.657829	.666666	.746031
.5000	.615864	.616504	.557459	.481589	.500005	.659574
.6666	.525994	.527610	.440077	.306795	.334474	.589743
.8333	[illegible]	.452446	.339593	.152269	.181349	.532153
1.0000	.365256	.388647	.256027	.054872	.073554	.463971
1.1666	.326095	.334313	.188595	.016122	.024054	[illegible]
1.3333	.278261	.287097	.135814	.004507	.007439	.407407
1.5000	.232586	.248131	.095721			
1.6666	.194093	.213973	.066132			
1.8333	.160059	.184554	.044680			
2.0000	.129968	.159157	.029992			
2.1666	.103502	.137181	.019793			
2.3333	.080530	.118122	.012941			
2.5000	.061036	.101561	.008408			
2.6666	.045009	.087144	.005446			
2.8333	.032325	.074576				
3.0000	.022683	.063610				

<u>**TABLE 12**</u>

<u>**Mean Number of Units in the System**</u> (page 121)

	Five Server	Reneging	b=-1, c=-1	b=0, c=0	b=-1, c=0
			Mean Number of Units		
.1666	.166666	.189991	.209015	.205767	.145911
.3333	.333336	.430403	.549514	.550324	.250264
.5000	.500035	.729837	1.161745	1.094398	.353226
.6666	.666843	1.093641	2.403758	2.208139	.431058
.8333	.833929	1.325035	4.832079	5.407185	.497829
1.0000	1.001581	2.024083	8.017306	7.494484	.556296
1.1666	1.170255	2.587536	10.511380	10.121503	.60839?
1.3333	1.340497	3.209014	11.977313	11.740022	.655542
1.5000	1.513164	3.679362			
1.6666	1.689252	4.587031			.738364
1.8333	1.870025	5.318116			
2.0000	2.057044	6.057066			.812001
2.1666	2.252187	66787653			
2.3333	2.457686	7.494426			.878449
2.5000	2.676169	8.164295			
2.6666	2.910601	8.787631			.940650
2.8333	3.440462				
3.0000	3.742623				1.000098

TABLE 13

Probability of No Delay (page 144)

	Reneging	No balking & reneging	Partial reneging
		Probability of No Delay	
.1666	.898251	.837697	.837697
.3333	.697558	.694692	.694725
.5000	.582545	.576921	.576422
.6666	.490478	.480298	.480948
.8333	.416835	.403018	.404400
1.0000	.357[illegible]	.340500	.342882
1.1666	.309850	.289529	.293121
1.3333	.270500	.247589	.252536
1.5000	.237878	.212754	.219142
1.6666	.210560	.183558	.191425
[illegible]	.187460	.158880	.168224
2.0000	.167785	.137861	.148650
2.1666	.150867	.119839	. 13201
2.3333	.136228	.104296	.117774
2.5000	.123477	.090827	.105511
2.6666	.112304	.079110	.094887
2.8333	.102461	.068888	.085633
3.0000	.09[illegible]746	.059954	.077531
3.1666	.085996	.052137	.070405
3.3333	.079074	.045297	.064110
3.5000	.072869	.039315	.058526
3.6666	.067295	.034088	.053555
3.8333	.062245	.029527	.049113
4.0000	.057682	.025555	.045132

TABLE 14

Mean Number of Units in the System (page 145)

	Reneging	No balking & reneging	Partial reneging
		Mean Number of Units in the System	
.1666	.027247	.028874	·028868
.3333	.096404	.107699	·107365
.5000	.190411	.223277	·220655
.6666	.297654	.365331	·355479
.8333	.411321	.527584	·502225
1.0000	.527078	.706815	·654575
1.1666	.644671	.901659	·808632
1.3333	.761590	1.112885	·9620[illegible]
1.5000	.877716	1.340969	1·113631
1.6666	.992981	1.587738	1·262607
1.8333	1.107[illegible]8	1.855167	1·408722
2[illegible]	1.220981	2.145341	1·551921
2[illegible]	1.333899	2.460236	1·692262
2.3333	1.446255	2.801517	1·829959
2.5000	1.558176	3.170310	1·965142
2.6666	1.669781	3.567038	2·098044
2.8333	1.781188	3.991273	2·228678
3.0000	1.892500	4.441664	2·357852
3.1666	2.003812	4.915930	2·485164
3.3333	2.115211	5.410935	2·611001
3.5000	2.226774	5.922813	2·735538
3.6666	2.338571	6.447176	2·858951
3.8333	2.450658	6.979325	2·981336
4.00000	2.563099	7.514477	3·102886

TABLE 15

Mean Number of Units in the System (page 146)

	No balking and remeging	balking and partial remeging	balking	remeging	balking and remeging
	Mean Number of Units in the System				
.1666	.0271	.0269	.0269	.0256	.0255
.3333	.1096	.1008	.1009		.0916
.5000	.2153	.2084	.2095	.1840	.1825
.6666	.3553	.3372	.3417	.2919	.2869
.8333	.5161	.4777	.4897	.4061	.3974
1.0000	.6941	.6231	.6485	.7598	.5099
1.1666	.8679	.7695	.8146	.8771	.6222
1.3333	1.0974	.91428	.9868	.9934	.7329
1.5000	1.3232	1.0559	1.1637	1.1088	.8417
1.6666	1.56735	1.1938	1.3446	1.2231	.9483
1.83333	1.8314	1.3275	1.5292	1.3366	1.0527
2.0000	2.1177	1.4571	1.7172	1.4495	1.1550
[illegible]	2.5482	1.5825	1.9083	1.5617	1.2552
[illegible]	2.7648	1.7040	2.1024	1.6736	1.3536
[illegible]	[illegible]	1.8218	2.2992	1.7851	1.4502
2.6066	[illegible]	1.9361	2.4987	1.8965	1.5452
2.8333	3.9419	2.0471	2.7005	2.0077	1.6388
3.0000	4.3880	2.1551	2.9044	2.1190	1.7310
3.1666	4.8597	2.2602	3.1103	2.2304	1.8219
3.3333	5.3533	2.3628	3.3177	2.3419	1.9117
3.5000	5.8649	2.4628	3.5265	2.4527	2.0004
3.6166	5.3903	2.5607	3.7363	2.5658	2.0880
3.83333	6.9246	2.6563	3.9469	2.6783	2.1747
4.0000	7.4630	2.7501	4.1580	2.7911	2.2805

TABLE 16

Probability of No Delay (page 155)

	N = 5	N = 10	N = 15	N = 20	N = 25
			Probability of No Delay		
.1694	.83052773	.83050848	.83050846	.83050846	.83050847
.3389	.66200556	.66102144	.66101697	.66101692	.66101694
.5084	.50007261	.49181431	.49153525	.49152574	.49152543
.6779	.35639081	.32657570	.32267673	.32212581	.32204706
.8474	.24178010	.18201218	.16416108	.15741222	.15563344
1.0169	.15900883	.08546046	.05493295	.04004437	.03092725
1.1864	.10332619	.03354631	.01293599	.00529086	
1.3559	.06751583	.01294592	.00274725		
1.5254	.04436240	.00509463			
1.6949	.02971470				
1.8644	.02026701				
2.0338	.01407955				
2.2033	.00[illegible]95675				
[illegible]	.00[illegible]16070				

TABLE 17

Mean Number of Units in the System (page 156)

	N = 5	N = 10	N = 15	N = 20	N = 25
	Mean Number of Units in the System				
.1694	.0344	.0345	.0345	.0345	.0345
.3389	.16657	.1737	.1738	.1738	.1743
.5084	.4358	.5200	.5257	.5259	.5253
.6779	.8114	1.2799	1.3964	1.4214	1.4262
.8474	1.3269	2.6318	3.5087	4.0457	4.3551
1.0169	1.8171	4.3150	6.9557	9.6899	12.5036
1.1864	2.2595	5.7746	9.8728	14.3457	
1.3559	2.6341	6.7992	11.5068		
1.5254	2.9418	7.47116			
1.6949	3.1919				
[illegible]	3.3955				
[illegible]	3.5527				
[illegible]	3.7013				
2.3728	3.8173				

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